

The Product Space of Real Normed Spaces and its Properties

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Summary. In this article, we define the product space of real linear spaces and real normed spaces. We also describe properties of these spaces.

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The terminology and notation used here are introduced in the following articles: [20], [9], [22], [2], [1], [19], [5], [23], [7], [10], [8], [4], [13], [12], [21], [14], [3], [6], [16], [11], [15], [17], and [18].

1. THE PRODUCT SPACE OF REAL LINEAR SPACES

The following propositions are true:

- (1) Let s, t be sequences of real numbers and g be a real number. Suppose that for every element n of \mathbb{N} holds $t(n) = |s(n) - g|$. Then s is convergent and $\lim s = g$ if and only if t is convergent and $\lim t = 0$.
- (2) Let x, y be finite sequences of elements of \mathbb{R} . Suppose $\text{len } x = \text{len } y$ and for every element i of \mathbb{N} such that $i \in \text{Seg len } x$ holds $0 \leq x(i)$ and $x(i) \leq y(i)$. Then $|x| \leq |y|$.
- (3) Let F be a finite sequence of elements of \mathbb{R} . If for every element i of \mathbb{N} such that $i \in \text{dom } F$ holds $F(i) = 0$, then $\sum F = 0$.

Let f be a function and let X be a set. A function is called a multi-operation of X and f if:

(Def. 1) $\text{dom } f = \text{dom } f$ and for every set i such that $i \in \text{dom } f$ holds $f(i)$ is a function from $\{X, f(i)\}$ into $f(i)$.

Let F be a sequence of non empty sets and let X be a set. Observe that every multi-operation of X and F is finite sequence-like.

We now state the proposition

(4) Let X be a set, F be a sequence of non empty sets, and p be a finite sequence. Then p is a multi-operation of X and F if and only if $\text{len } p = \text{len } F$ and for every set i such that $i \in \text{dom } F$ holds $p(i)$ is a function from $\{X, F(i)\}$ into $F(i)$.

Let F be a sequence of non empty sets, let X be a set, let p be a multi-operation of X and F , and let i be an element of $\text{dom } F$. Then $p(i)$ is a function from $\{X, F(i)\}$ into $F(i)$.

Next we state the proposition

(5) Let X be a non empty set, F be a sequence of non empty sets, and f, g be functions from $\{X, \prod F\}$ into $\prod F$. Suppose that for every element x of X and for every element d of $\prod F$ and for every element i of $\text{dom } F$ holds $f(x, d)(i) = g(x, d)(i)$. Then $f = g$.

Let F be a sequence of non empty sets, let X be a non empty set, and let p be a multi-operation of X and F . The functor $\prod^\circ p$ yielding a function from $\{X, \prod F\}$ into $\prod F$ is defined as follows:

(Def. 2) For every element x of X and for every element d of $\prod F$ and for every element i of $\text{dom } F$ holds $(\prod^\circ p)(x, d)(i) = p(i)(x, d(i))$.

Let R be a binary relation. We say that R is real-linear-space-yielding if and only if:

(Def. 3) For every set S such that $S \in \text{rng } R$ holds S is a real linear space.

Let us note that there exists a finite sequence which is non empty and real-linear-space-yielding.

A real linear space-sequence is a non empty real-linear-space-yielding finite sequence.

Let G be a real linear space-sequence and let j be an element of $\text{dom } G$. Then $G(j)$ is a real linear space.

Let G be a real linear space-sequence. The functor \overline{G} yielding a sequence of non empty sets is defined by:

(Def. 4) $\text{len } \overline{G} = \text{len } G$ and for every element j of $\text{dom } G$ holds $\overline{G}(j) =$ the carrier of $G(j)$.

Let G be a real linear space-sequence and let j be an element of $\text{dom } \overline{G}$. Then $G(j)$ is a real linear space.

Let G be a real linear space-sequence. The functor $\langle +_{G_i} \rangle_i$ yielding a family of binary operations of \overline{G} is defined as follows:

(Def. 5) $\text{len}(\langle +_{G_i} \rangle_i) = \text{len } \overline{G}$ and for every element j of $\text{dom } \overline{G}$ holds $\langle +_{G_i} \rangle_i(j) =$ the addition of $G(j)$.

The functor $\langle -_{G_i} \rangle_i$ yields a family of unary operations of \overline{G} and is defined as follows:

(Def. 6) $\text{len}(\langle -_{G_i} \rangle_i) = \text{len } \overline{G}$ and for every element j of $\text{dom } \overline{G}$ holds $\langle -_{G_i} \rangle_i(j) =$ $\text{comp } G(j)$.

The functor $\langle 0_{G_i} \rangle_i$ yielding an element of $\prod \overline{G}$ is defined by:

(Def. 7) For every element j of $\text{dom } \overline{G}$ holds $\langle 0_{G_i} \rangle_i(j) =$ the zero of $G(j)$.

The functor $\text{multop } G$ yields a multi-operation of \mathbb{R} and \overline{G} and is defined by:

(Def. 8) $\text{len } \text{multop } G = \text{len } \overline{G}$ and for every element j of $\text{dom } \overline{G}$ holds $(\text{multop } G)(j) =$ the external multiplication of $G(j)$.

Let G be a real linear space-sequence. The functor $\prod G$ yielding a strict non empty RLS structure is defined by:

(Def. 9) $\prod G = \langle \prod \overline{G}, \langle 0_{G_i} \rangle_i, \prod^\circ(\langle +_{G_i} \rangle_i), \prod^\circ \text{multop } G \rangle$.

Let G be a real linear space-sequence. One can check that $\prod G$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

2. THE PRODUCT SPACE OF REAL NORMED SPACES

Let R be a binary relation. We say that R is real-norm-space-yielding if and only if:

(Def. 10) For every set x such that $x \in \text{rng } R$ holds x is a real normed space.

One can check that there exists a finite sequence which is non empty and real-norm-space-yielding.

A real norm space-sequence is a non empty real-norm-space-yielding finite sequence.

Let G be a real norm space-sequence and let j be an element of $\text{dom } G$. Then $G(j)$ is a real normed space.

Let us note that every finite sequence which is real-norm-space-yielding is also real-linear-space-yielding.

Let G be a real norm space-sequence and let x be an element of $\prod \overline{G}$. The functor $\text{normsequence}(G, x)$ yields an element of $\mathcal{R}^{\text{len } G}$ and is defined as follows:

(Def. 11) $\text{len } \text{normsequence}(G, x) = \text{len } G$ and for every element j of $\text{dom } G$ holds $(\text{normsequence}(G, x))(j) =$ (the norm of $G(j))(x(j))$.

Let G be a real norm space-sequence. The functor $\text{productnorm } G$ yields a function from $\prod (\overline{G} \text{ qua real linear space-sequence})$ into \mathbb{R} and is defined by:

(Def. 12) For every element x of $\prod \overline{G}$ holds $(\text{productnorm } G)(x) = |\text{normsequence}(G, x)|$.

Let G be a real norm space-sequence. The functor $\prod G$ yielding a strict non empty normed structure is defined as follows:

(Def. 13) The RLS structure of $\prod G = \prod(G \text{ qua real linear space-sequence})$ and the norm of $\prod G = \text{productnorm } G$.

In the sequel G is a real norm space-sequence.

We now state four propositions:

- (6) $\prod G = \langle \prod \overline{G}, \langle 0_{G_i} \rangle_i, \prod^\circ(\langle +_{G_i} \rangle_i), \prod^\circ \text{ multop } G, \text{productnorm } G \rangle$.
- (7) For every vector x of $\prod G$ and for every element y of $\prod \overline{G}$ such that $x = y$ holds $\|x\| = |\text{normsequence}(G, y)|$.
- (8) For all elements x, y, z of $\prod \overline{G}$ and for every element i of \mathbb{N} such that $i \in \text{dom } x$ and $z = (\prod^\circ(\langle +_{G_i} \rangle_i))(x, y)$ holds $(\text{normsequence}(G, z))(i) \leq (\text{normsequence}(G, x) + \text{normsequence}(G, y))(i)$.
- (9) For every element x of $\prod \overline{G}$ and for every element i of \mathbb{N} such that $i \in \text{dom } x$ holds $0 \leq (\text{normsequence}(G, x))(i)$.

Let G be a real norm space-sequence. Observe that $\prod G$ is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

- (10) Let G be a real norm space-sequence, i be an element of $\text{dom } G$, x be a point of $\prod G$, y be an element of $\prod \overline{G}$, and x_1 be a point of $G(i)$. If $y = x$ and $x_1 = y(i)$, then $\|x_1\| \leq \|x\|$.
- (11) Let G be a real norm space-sequence, i be an element of $\text{dom } G$, x, y be points of $\prod G$, x_1, y_1 be points of $G(i)$, and z_1, z_2 be elements of $\prod \overline{G}$. If $x_1 = z_1(i)$ and $z_1 = x$ and $y_1 = z_2(i)$ and $z_2 = y$, then $\|y_1 - x_1\| \leq \|y - x\|$.
- (12) Let G be a real norm space-sequence, s_1 be a sequence of $\prod G$, x_0 be a point of $\prod G$, and y_0 be an element of $\prod \overline{G}$. Suppose $x_0 = y_0$ and s_1 is convergent and $\lim s_1 = x_0$. Let i be an element of $\text{dom } G$. Then there exists a sequence s_2 of $G(i)$ such that s_2 is convergent and $y_0(i) = \lim s_2$ and for every element m of \mathbb{N} there exists an element s_3 of $\prod \overline{G}$ such that $s_3 = s_1(m)$ and $s_2(m) = s_3(i)$.
- (13) Let G be a real norm space-sequence, s_1 be a sequence of $\prod G$, x_0 be a point of $\prod G$, and y_0 be an element of $\prod \overline{G}$. Suppose that
 - (i) $x_0 = y_0$, and
 - (ii) for every element i of $\text{dom } G$ there exists a sequence s_2 of $G(i)$ such that s_2 is convergent and $y_0(i) = \lim s_2$ and for every element m of \mathbb{N} there exists an element s_3 of $\prod \overline{G}$ such that $s_3 = s_1(m)$ and $s_2(m) = s_3(i)$.
 Then s_1 is convergent and $\lim s_1 = x_0$.
- (14) For every real norm space-sequence G such that for every element i of $\text{dom } G$ holds $G(i)$ is complete holds $\prod G$ is complete.

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