

String Rewriting Systems

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Summary. Basing on the definitions from [15], semi-Thue systems, Thue systems, and direct derivations are introduced. Next, the standard reduction relation is defined that, in turn, is used to introduce derivations using the theory from [1]. Finally, languages generated by rewriting systems are defined as all strings reachable from an initial word. This is followed by the introduction of the equivalence of semi-Thue systems with respect to the initial word.

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The notation and terminology used here are introduced in the following papers: [11], [13], [8], [16], [10], [4], [17], [14], [7], [18], [2], [1], [3], [12], [5], [6], and [9].

1. PRELIMINARIES

We adopt the following convention: x denotes a set, k, l denote natural numbers, and p, q denote finite sequences.

Next we state two propositions:

- (1) If $k \notin \text{dom } p$ and $k + 1 \in \text{dom } p$, then $k = 0$.
- (2) If $k > \text{len } p$ and $k \leq \text{len}(p \hat{\ } q)$, then there exists l such that $k = \text{len } p + l$ and $l \geq 1$ and $l \leq \text{len } q$.

In the sequel R denotes a binary relation and p, q denote reduction sequences w.r.t. R .

Next we state two propositions:

- (3) If $k \geq 1$, then $p \upharpoonright k$ is a reduction sequence w.r.t. R .
- (4) If $k \in \text{dom } p$, then there exists q such that $\text{len } q = k$ and $q(1) = p(1)$ and $q(\text{len } q) = p(k)$.

2. FINITE 0-SEQUENCE YIELDING FUNCTIONS AND FINITE SEQUENCES

Let f be a function. We say that f is finite-0-sequence-yielding if and only if:

(Def. 1) If $x \in \text{dom } f$, then $f(x)$ is a finite 0-sequence.

Let us mention that \emptyset is finite-0-sequence-yielding.

Let f be a finite 0-sequence. Observe that $\langle f \rangle$ is finite-0-sequence-yielding.

Let us observe that there exists a function which is finite-0-sequence-yielding.

Let p be a finite-0-sequence-yielding function and let us consider x . Then $p(x)$ is a finite 0-sequence.

One can verify that there exists a finite sequence which is finite-0-sequence-yielding.

Let E be a set. Note that every finite sequence of elements of E^ω is finite-0-sequence-yielding.

Let p, q be finite-0-sequence-yielding finite sequences. Observe that $p \hat{\ } q$ is finite-0-sequence-yielding.

3. CONCATENATION OF A FINITE 0-SEQUENCE WITH ALL ELEMENTS OF A FINITE 0-SEQUENCE YIELDING FUNCTION

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding function.

The functor $s + p$ yields a finite-0-sequence-yielding function and is defined by:

(Def. 2) $\text{dom}(s + p) = \text{dom } p$ and for every x such that $x \in \text{dom } p$ holds $(s + p)(x) = s \hat{\ } p(x)$.

The functor $p + s$ yielding a finite-0-sequence-yielding function is defined by:

(Def. 3) $\text{dom}(p + s) = \text{dom } p$ and for every x such that $x \in \text{dom } p$ holds $(p + s)(x) = p(x) \hat{\ } s$.

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding finite sequence. Note that $s + p$ is finite sequence-like and $p + s$ is finite sequence-like.

We adopt the following convention: E denotes a set, s, t denote finite 0-sequences, and p, q denote finite-0-sequence-yielding finite sequences.

The following propositions are true:

$$(5) \quad \text{len}(s + p) = \text{len } p \text{ and } \text{len}(p + s) = \text{len } p.$$

$$(6) \quad \langle \rangle_E + p = p \text{ and } p + \langle \rangle_E = p.$$

$$(7) \quad s + (t + p) = s \hat{\ } t + p \text{ and } p + t + s = p + t \hat{\ } s.$$

$$(8) \quad s + (p + t) = (s + p) + t.$$

$$(9) \quad s + p \hat{\ } q = (s + p) \hat{\ } (s + q) \text{ and } p \hat{\ } q + s = (p + s) \hat{\ } (q + s).$$

4. SEMI-THUE SYSTEMS AND THUE SYSTEMS

Let E be a set, let p be a finite sequence of elements of E^ω , and let k be a natural number. Then $p(k)$ is an element of E^ω .

Let E be a set, let k be a natural number, and let s be an element of E^ω . Then $k \mapsto s$ is a finite sequence of elements of E^ω .

Let E be a set, let s be an element of E^ω , and let p be a finite sequence of elements of E^ω . Then $s + p$ is a finite sequence of elements of E^ω . Then $p + s$ is a finite sequence of elements of E^ω .

Let E be a set. A semi-Thue-system of E is a binary relation on E^ω .

In the sequel E is a set and S, T, U are semi-Thue-systems of E .

Let S be a binary relation. Observe that $S \cup S^\sim$ is symmetric.

Let us consider E . One can check that there exists a semi-Thue-system of E which is symmetric.

Let E be a set. A Thue-system of E is a symmetric semi-Thue-system of E .

5. DIRECT DERIVATIONS

We follow the rules: s, t, s_1, t_1, u, v, w are elements of E^ω and p is a finite sequence of elements of E^ω .

Let us consider E, S, s, t . The predicate $s \rightarrow_S t$ is defined by:

(Def. 4) $\langle s, t \rangle \in S$.

Let us consider E, S, s, t . The predicate $s \Rightarrow_S t$ is defined as follows:

(Def. 5) There exist v, w, s_1, t_1 such that $s = v \wedge s_1 \wedge w$ and $t = v \wedge t_1 \wedge w$ and $s_1 \rightarrow_S t_1$.

The following propositions are true:

- (10) If $s \rightarrow_S t$, then $s \Rightarrow_S t$.
- (11) If $s \Rightarrow_S s$, then there exist v, w, s_1 such that $s = v \wedge s_1 \wedge w$ and $s_1 \rightarrow_S s_1$.
- (12) If $s \Rightarrow_S t$, then $u \wedge s \Rightarrow_S u \wedge t$ and $s \wedge u \Rightarrow_S t \wedge u$.
- (13) If $s \Rightarrow_S t$, then $u \wedge s \wedge v \Rightarrow_S u \wedge t \wedge v$.
- (14) If $s \rightarrow_S t$, then $u \wedge s \Rightarrow_S u \wedge t$ and $s \wedge u \Rightarrow_S t \wedge u$.
- (15) If $s \rightarrow_S t$, then $u \wedge s \wedge v \Rightarrow_S u \wedge t \wedge v$.
- (16) If S is a Thue-system of E and $s \rightarrow_S t$, then $t \rightarrow_S s$.
- (17) If S is a Thue-system of E and $s \Rightarrow_S t$, then $t \Rightarrow_S s$.
- (18) If $S \subseteq T$ and $s \rightarrow_S t$, then $s \rightarrow_T t$.
- (19) If $S \subseteq T$ and $s \Rightarrow_S t$, then $s \Rightarrow_T t$.
- (20) $s \not\Rightarrow_{\emptyset_{E^\omega, E^\omega}} t$.
- (21) If $s \Rightarrow_{S \cup T} t$, then $s \Rightarrow_S t$ or $s \Rightarrow_T t$.

6. REDUCTION RELATION

Let us consider E . Then id_E is a binary relation on E .

Let us consider E, S . The functor \Rightarrow_S yielding a binary relation on E^ω is defined as follows:

(Def. 6) $\langle s, t \rangle \in \Rightarrow_S$ iff $s \Rightarrow_S t$.

The following propositions are true:

- (22) $S \subseteq \Rightarrow_S$.
- (23) Suppose p is a reduction sequence w.r.t. \Rightarrow_S . Then $p + u$ is a reduction sequence w.r.t. \Rightarrow_S and $u + p$ is a reduction sequence w.r.t. \Rightarrow_S .
- (24) If p is a reduction sequence w.r.t. \Rightarrow_S , then $(t + p) + u$ is a reduction sequence w.r.t. \Rightarrow_S .
- (25) If S is a Thue-system of E , then $\Rightarrow_S = (\Rightarrow_S)^\smile$.
- (26) If $S \subseteq T$, then $\Rightarrow_S \subseteq \Rightarrow_T$.
- (27) $\Rightarrow_{\text{id}_{E^\omega}} = \text{id}_{E^\omega}$.
- (28) $\Rightarrow_{S \cup \text{id}_{E^\omega}} = \Rightarrow_S \cup \text{id}_{E^\omega}$.
- (29) $\Rightarrow_{\emptyset_{E^\omega, E^\omega}} = \emptyset_{E^\omega, E^\omega}$.
- (30) If $s \Rightarrow_{\Rightarrow_S} t$, then $s \Rightarrow_S t$.
- (31) $\Rightarrow_{\Rightarrow_S} = \Rightarrow_S$.

7. DERIVATIONS

Let us consider E, S, s, t . The predicate $s \Rightarrow_S^* t$ is defined by:

(Def. 7) \Rightarrow_S reduces s to t .

One can prove the following propositions:

- (32) $s \Rightarrow_S^* s$.
- (33) If $s \Rightarrow_S t$, then $s \Rightarrow_S^* t$.
- (34) If $s \rightarrow_S t$, then $s \Rightarrow_S^* t$.
- (35) If $s \Rightarrow_S^* t$ and $t \Rightarrow_S^* u$, then $s \Rightarrow_S^* u$.
- (36) If $s \Rightarrow_S^* t$, then $s \wedge u \Rightarrow_S^* t \wedge u$ and $u \wedge s \Rightarrow_S^* u \wedge t$.
- (37) If $s \Rightarrow_S^* t$, then $u \wedge s \wedge v \Rightarrow_S^* u \wedge t \wedge v$.
- (38) If $s \Rightarrow_S^* t$ and $u \Rightarrow_S^* v$, then $s \wedge u \Rightarrow_S^* t \wedge v$ and $u \wedge s \Rightarrow_S^* v \wedge t$.
- (39) If S is a Thue-system of E and $s \Rightarrow_S^* t$, then $t \Rightarrow_S^* s$.
- (40) If $S \subseteq T$ and $s \Rightarrow_S^* t$, then $s \Rightarrow_T^* t$.
- (41) $s \Rightarrow_S^* t$ iff $s \Rightarrow_{S \cup \text{id}_{E^\omega}}^* t$.
- (42) If $s \Rightarrow_{\emptyset_{E^\omega, E^\omega}}^* t$, then $s = t$.
- (43) If $s \Rightarrow_{\Rightarrow_S}^* t$, then $s \Rightarrow_S^* t$.

- (44) If $s \Rightarrow_S^* t$ and $u \Rightarrow_{\{\langle s, t \rangle\}} v$, then $u \Rightarrow_S^* v$.
- (45) If $s \Rightarrow_S^* t$ and $u \Rightarrow_{S \cup \{\langle s, t \rangle\}}^* v$, then $u \Rightarrow_S^* v$.

8. LANGUAGES GENERATED BY SEMI-THUE SYSTEMS

Let us consider E, S, w . The functor $\text{Lang}(w, S)$ yields a subset of E^ω and is defined by:

(Def. 8) $\text{Lang}(w, S) = \{s : w \Rightarrow_S^* s\}$.

Next we state two propositions:

- (46) $s \in \text{Lang}(w, S)$ iff $w \Rightarrow_S^* s$.
- (47) $w \in \text{Lang}(w, S)$.

Let E be a non empty set, let S be a semi-Thue-system of E , and let w be an element of E^ω . Note that $\text{Lang}(w, S)$ is non empty.

We now state four propositions:

- (48) If $S \subseteq T$, then $\text{Lang}(w, S) \subseteq \text{Lang}(w, T)$.
- (49) $\text{Lang}(w, S) = \text{Lang}(w, S \cup \text{id}_{E^\omega})$.
- (50) $\text{Lang}(w, \emptyset_{E^\omega, E^\omega}) = \{w\}$.
- (51) $\text{Lang}(w, \text{id}_{E^\omega}) = \{w\}$.

9. EQUIVALENCE OF SEMI-THUE SYSTEMS

Let us consider E, S, T, w . We say that S and T are equivalent wrt w if and only if:

(Def. 9) $\text{Lang}(w, S) = \text{Lang}(w, T)$.

The following propositions are true:

- (52) S and S are equivalent wrt w .
- (53) If S and T are equivalent wrt w , then T and S are equivalent wrt w .
- (54) Suppose S and T are equivalent wrt w and T and U are equivalent wrt w . Then S and U are equivalent wrt w .
- (55) S and $S \cup \text{id}_{E^\omega}$ are equivalent wrt w .
- (56) Suppose S and T are equivalent wrt w and $S \subseteq U$ and $U \subseteq T$. Then S and U are equivalent wrt w and U and T are equivalent wrt w .
- (57) S and \Rightarrow_S are equivalent wrt w .
- (58) If S and T are equivalent wrt w and $\Rightarrow_{S \cup T}$ reduces w to s , then \Rightarrow_S reduces w to s .
- (59) If S and T are equivalent wrt w and $w \Rightarrow_{S \cup T}^* s$, then $w \Rightarrow_S^* s$.
- (60) If S and T are equivalent wrt w , then S and $S \cup T$ are equivalent wrt w .

- (61) If $s \Rightarrow_S t$, then S and $S \cup \{\langle s, t \rangle\}$ are equivalent wrt w .
 (62) If $s \Rightarrow_S^* t$, then S and $S \cup \{\langle s, t \rangle\}$ are equivalent wrt w .

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