FORMALIZED MATHEMATICS
2007, Vol. **15**, No. **3**, Pages 121–126
DOI: 10.2478/v10037-007-0013-8

String Rewriting Systems

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Summary. Basing on the definitions from [15], semi-Thue systems, Thue systems, and direct derivations are introduced. Next, the standard reduction relation is defined that, in turn, is used to introduce derivations using the theory from [1]. Finally, languages generated by rewriting systems are defined as all strings reachable from an initial word. This is followed by the introduction of the equivalence of semi-Thue systems with respect to the initial word.

MML identifier: `REWRITE2`, version: `7.8.05 4.87.985`

The notation and terminology used here are introduced in the following papers: [11], [13], [8], [16], [10], [4], [17], [14], [7], [18], [2], [1], [3], [12], [5], [6], and [9].

1. Preliminaries

We adopt the following convention: x denotes a set, k, l denote natural numbers, and p, q denote finite sequences.

Next we state two propositions:

(1) If $k \notin \operatorname{dom} p$ and $k+1 \in \operatorname{dom} p$, then $k = 0$.

(2) If $k > \operatorname{len} p$ and $k \leq \operatorname{len}(p \smallfrown q)$, then there exists l such that $k = \operatorname{len} p + l$ and $l \geq 1$ and $l \leq \operatorname{len} q$.

In the sequel R denotes a binary relation and p, q denote reduction sequences w.r.t. R.

Next we state two propositions:

(3) If $k \geq 1$, then $p {\restriction} k$ is a reduction sequence w.r.t. R.

(4) If $k \in \operatorname{dom} p$, then there exists q such that $\operatorname{len} q = k$ and $q(1) = p(1)$ and $q(\operatorname{len} q) = p(k)$.

ISSN 1426–2630

2. Finite 0-sequence Yielding Functions and Finite Sequences

Let f be a function. We say that f is finite-0-sequence-yielding if and only if:

(Def. 1) If $x \in \operatorname{dom} f$, then $f(x)$ is a finite 0-sequence.

Let us mention that $\emptyset$ is finite-0-sequence-yielding.

Let f be a finite 0-sequence. Observe that $\langle f \rangle$ is finite-0-sequence-yielding.

Let us observe that there exists a function which is finite-0-sequence-yielding.

Let p be a finite-0-sequence-yielding function and let us consider x. Then $p(x)$ is a finite 0-sequence.

One can verify that there exists a finite sequence which is finite-0-sequence-yielding.

Let E be a set. Note that every finite sequence of elements of E^ω is finite-0-sequence-yielding.

Let p, q be finite-0-sequence-yielding finite sequences. Observe that $p \frown q$ is finite-0-sequence-yielding.

3. Concatenation of a Finite 0-sequence with All Elements of a Finite 0-sequence Yielding Function

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding function. The functor $s + p$ yields a finite-0-sequence-yielding function and is defined by:

(Def. 2) $\operatorname{dom}(s + p) = \operatorname{dom} p$ and for every x such that $x \in \operatorname{dom} p$ holds $(s + p)(x) = s \frown p(x)$.

The functor $p + s$ yielding a finite-0-sequence-yielding function is defined by:

(Def. 3) $\operatorname{dom}(p + s) = \operatorname{dom} p$ and for every x such that $x \in \operatorname{dom} p$ holds $(p + s)(x) = p(x) \frown s$.

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding finite sequence. Note that $s + p$ is finite sequence-like and $p + s$ is finite sequence-like.

We adopt the following convention: E denotes a set, s, t denote finite 0-sequences, and p, q denote finite-0-sequence-yielding finite sequences.

The following propositions are true:

(5) $\operatorname{len}(s + p) = \operatorname{len} p$ and $\operatorname{len}(p + s) = \operatorname{len} p$.

(6) $\langle\rangle_E + p = p$ and $p + \langle\rangle_E = p$.

(7) $s + (t + p) = s \frown t + p$ and $p + t + s = p + t \frown s$.

(8) $s + (p + t) = (s + p) + t$.

(9) $s + p \frown q = (s + p) \frown (s + q)$ and $p \frown q + s = (p + s) \frown (q + s)$.

4. Semi-Thue Systems and Thue Systems

Let E be a set, let p be a finite sequence of elements of E^ω, and let k be a natural number. Then $p(k)$ is an element of E^ω.

Let E be a set, let k be a natural number, and let s be an element of E^ω. Then $k \mapsto s$ is a finite sequence of elements of E^ω.

Let E be a set, let s be an element of E^ω, and let p be a finite sequence of elements of E^ω. Then $s + p$ is a finite sequence of elements of E^ω. Then $p + s$ is a finite sequence of elements of E^ω.

Let E be a set. A semi-Thue-system of E is a binary relation on E^ω.

In the sequel E is a set and S, T, U are semi-Thue-systems of E.

Let S be a binary relation. Observe that $S \cup S^\smile$ is symmetric.

Let us consider E. One can check that there exists a semi-Thue-system of E which is symmetric.

Let E be a set. A Thue-system of E is a symmetric semi-Thue-system of E.

5. Direct Derivations

We follow the rules: s, t, s_1, t_1, u, v, w are elements of E^ω and p is a finite sequence of elements of E^ω.

Let us consider E, S, s, t. The predicate $s \to_S t$ is defined by:

(Def. 4) $\langle s, t\rangle \in S$.

Let us consider E, S, s, t. The predicate $s \Rightarrow_S t$ is defined as follows:

(Def. 5) There exist v, w, s_1, t_1 such that $s = v \frown s_1 \frown w$ and $t = v \frown t_1 \frown w$ and $s_1 \to_S t_1$.

The following propositions are true:

(10) If $s \to_S t$, then $s \Rightarrow_S t$.

(11) If $s \Rightarrow_S s$, then there exist v, w, s_1 such that $s = v \frown s_1 \frown w$ and $s_1 \to_S s_1$.

(12) If $s \Rightarrow_S t$, then $u \frown s \Rightarrow_S u \frown t$ and $s \frown u \Rightarrow_S t \frown u$.

(13) If $s \Rightarrow_S t$, then $u \frown s \frown v \Rightarrow_S u \frown t \frown v$.

(14) If $s \to_S t$, then $u \frown s \Rightarrow_S u \frown t$ and $s \frown u \Rightarrow_S t \frown u$.

(15) If $s \to_S t$, then $u \frown s \frown v \Rightarrow_S u \frown t \frown v$.

(16) If S is a Thue-system of E and $s \to_S t$, then $t \to_S s$.

(17) If S is a Thue-system of E and $s \Rightarrow_S t$, then $t \Rightarrow_S s$.

(18) If $S \subseteq T$ and $s \to_S t$, then $s \to_T t$.

(19) If $S \subseteq T$ and $s \Rightarrow_S t$, then $s \Rightarrow_T t$.

(20) $s \not\Rightarrow_{\emptyset_{E^\omega, E^\omega}} t$.

(21) If $s \Rightarrow_{S \cup T} t$, then $s \Rightarrow_S t$ or $s \Rightarrow_T t$.

6. Reduction Relation

Let us consider E. Then id_E is a binary relation on E.

Let us consider E, S. The functor $\Rightarrow_S$ yielding a binary relation on E^ω is defined as follows:

(Def. 6) $\langle s, t\rangle \in \Rightarrow_S$ iff $s \Rightarrow_S t$.

The following propositions are true:

(22) $S \subseteq \Rightarrow_S$.

(23) Suppose p is a reduction sequence w.r.t. $\Rightarrow_S$. Then $p + u$ is a reduction sequence w.r.t. $\Rightarrow_S$ and $u + p$ is a reduction sequence w.r.t. $\Rightarrow_S$.

(24) If p is a reduction sequence w.r.t. $\Rightarrow_S$, then $(t + p) + u$ is a reduction sequence w.r.t. $\Rightarrow_S$.

(25) If S is a Thue-system of E, then $\Rightarrow_S = (\Rightarrow_S)^\smile$.

(26) If $S \subseteq T$, then $\Rightarrow_S \subseteq \Rightarrow_T$.

(27) $\Rightarrow_{\mathrm{id}_{E^\omega}} = \mathrm{id}_{E^\omega}$.

(28) $\Rightarrow_{S \cup \mathrm{id}_{E^\omega}} = \Rightarrow_S \cup \mathrm{id}_{E^\omega}$.

(29) $\Rightarrow_{\emptyset_{E^\omega, E^\omega}} = \emptyset_{E^\omega, E^\omega}$.

(30) If $s \Rightarrow_{\Rightarrow_S} t$, then $s \Rightarrow_S t$.

(31) $\Rightarrow_{\Rightarrow_S} = \Rightarrow_S$.

7. Derivations

Let us consider E, S, s, t. The predicate $s \Rightarrow_S^* t$ is defined by:

(Def. 7) $\Rightarrow_S$ reduces s to t.

One can prove the following propositions:

(32) $s \Rightarrow_S^* s$.

(33) If $s \Rightarrow_S t$, then $s \Rightarrow_S^* t$.

(34) If $s \rightarrow_S t$, then $s \Rightarrow_S^* t$.

(35) If $s \Rightarrow_S^* t$ and $t \Rightarrow_S^* u$, then $s \Rightarrow_S^* u$.

(36) If $s \Rightarrow_S^* t$, then $s \frown u \Rightarrow_S^* t \frown u$ and $u \frown s \Rightarrow_S^* u \frown t$.

(37) If $s \Rightarrow_S^* t$, then $u \frown s \frown v \Rightarrow_S^* u \frown t \frown v$.

(38) If $s \Rightarrow_S^* t$ and $u \Rightarrow_S^* v$, then $s \frown u \Rightarrow_S^* t \frown v$ and $u \frown s \Rightarrow_S^* v \frown t$.

(39) If S is a Thue-system of E and $s \Rightarrow_S^* t$, then $t \Rightarrow_S^* s$.

(40) If $S \subseteq T$ and $s \Rightarrow_S^* t$, then $s \Rightarrow_T^* t$.

(41) $s \Rightarrow_S^* t$ iff $s \Rightarrow_{S \cup \mathrm{id}_{E^\omega}}^* t$.

(42) If $s \Rightarrow_{\emptyset_{E^\omega, E^\omega}}^* t$, then $s = t$.

(43) If $s \Rightarrow_{\Rightarrow_S}^* t$, then $s \Rightarrow_S^* t$.

(44) If $s \Rightarrow^*_S t$ and $u \Rightarrow_{\{\langle s,t\rangle\}} v$, then $u \Rightarrow^*_S v$.

(45) If $s \Rightarrow^*_S t$ and $u \Rightarrow^*_{S \cup \{\langle s,t\rangle\}} v$, then $u \Rightarrow^*_S v$.

8. Languages Generated by Semi-Thue Systems

Let us consider E, S, w. The functor $\mathrm{Lang}(w, S)$ yields a subset of E^ω and is defined by:

(Def. 8) $\mathrm{Lang}(w, S) = \{s : w \Rightarrow^*_S s\}$.

Next we state two propositions:

(46) $s \in \mathrm{Lang}(w, S)$ iff $w \Rightarrow^*_S s$.

(47) $w \in \mathrm{Lang}(w, S)$.

Let E be a non empty set, let S be a semi-Thue-system of E, and let w be an element of E^ω. Note that $\mathrm{Lang}(w, S)$ is non empty.

We now state four propositions:

(48) If $S \subseteq T$, then $\mathrm{Lang}(w, S) \subseteq \mathrm{Lang}(w, T)$.

(49) $\mathrm{Lang}(w, S) = \mathrm{Lang}(w, S \cup \mathrm{id}_{E^\omega})$.

(50) $\mathrm{Lang}(w, \emptyset_{E^\omega, E^\omega}) = \{w\}$.

(51) $\mathrm{Lang}(w, \mathrm{id}_{E^\omega}) = \{w\}$.

9. Equivalence of Semi-Thue Systems

Let us consider E, S, T, w. We say that S and T are equivalent wrt w if and only if:

(Def. 9) $\mathrm{Lang}(w, S) = \mathrm{Lang}(w, T)$.

The following propositions are true:

(52) S and S are equivalent wrt w.

(53) If S and T are equivalent wrt w, then T and S are equivalent wrt w.

(54) Suppose S and T are equivalent wrt w and T and U are equivalent wrt w. Then S and U are equivalent wrt w.

(55) S and $S \cup \mathrm{id}_{E^\omega}$ are equivalent wrt w.

(56) Suppose S and T are equivalent wrt w and $S \subseteq U$ and $U \subseteq T$. Then S and U are equivalent wrt w and U and T are equivalent wrt w.

(57) S and $\Rightarrow_S$ are equivalent wrt w.

(58) If S and T are equivalent wrt w and $\Rightarrow_{S \cup T}$ reduces w to s, then $\Rightarrow_S$ reduces w to s.

(59) If S and T are equivalent wrt w and $w \Rightarrow^*_{S \cup T} s$, then $w \Rightarrow^*_S s$.

(60) If S and T are equivalent wrt w, then S and $S \cup T$ are equivalent wrt w.

(61) If $s \Rightarrow_S t$, then S and $S \cup \{\langle s, t\rangle\}$ are equivalent wrt w.

(62) If $s \Rightarrow_S^* t$, then S and $S \cup \{\langle s, t\rangle\}$ are equivalent wrt w.

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Received July 17, 2007