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Determinant and Inverse of Matrices of Real Elements

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Summary. In this paper the classic theory of matrices of real elements (see e.g. [12], [13]) is developed. We prove selected equations that have been proved previously for matrices of field elements. Similarly, we introduce in this special context the determinant of a matrix, the identity and zero matrices, and the inverse matrix. The new concept discussed in the case of matrices of real numbers is the property of matrices as operators acting on finite sequences of real numbers from both sides. The relations of invertibility of matrices and the "onto" property of matrices as operators are discussed.

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The articles [24], [30], [9], [2], [22], [31], [7], [4], [5], [8], [3], [6], [28], [26], [21], [14], [29], [32], [23], [25], [27], [15], [34], [33], [19], [16], [11], [18], [20], [10], [17], [1], and [35] provide the terminology and notation for this paper.

1. Preliminaries

We use the following convention: D denotes a non empty set, k, n, m, i, j, l denote elements of $\mathbb{N}$, and K denotes a field.

We now state several propositions:

(1) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ and $x + y = \langle \underbrace{0, \ldots, 0}_{\operatorname{len} x} \rangle$ holds $x = -y$ and $y = -x$.

(2) Let A be a matrix over D and p be a finite sequence of elements of D. If $p = A(i)$ and $1 \leq i$ and $i \leq \operatorname{len} A$ and $1 \leq j$ and $j \leq \operatorname{width} A$ and $\operatorname{len} p = \operatorname{width} A$, then $A_{i,j} = p(j)$.

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(3) Let a be a real number and A be a matrix over $\mathbb{R}$. Suppose $\operatorname{len}(a \cdot A) = \operatorname{len} A$ and $\operatorname{width}(a \cdot A) = \operatorname{width} A$ and $\langle i, j\rangle \in$ the indices of A. Then $(a \cdot A)_{i,j} = a \cdot A_{i,j}$.

(4) For all matrices A, B over $\mathbb{R}$ of dimension n holds $\operatorname{len}(A \cdot B) = \operatorname{len} A$ and $\operatorname{width}(A \cdot B) = \operatorname{width} B$ and $\operatorname{len}(A \cdot B) = n$ and $\operatorname{width}(A \cdot B) = n$.

(5) For every real number a and for every matrix A over $\mathbb{R}$ holds $\operatorname{len}(a \cdot A) = \operatorname{len} A$ and $\operatorname{width}(a \cdot A) = \operatorname{width} A$.

2. Calculation of Matrices

We now state the proposition

(6) Let A, B be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. Then $\operatorname{len}(A - B) = \operatorname{len} A$ and $\operatorname{width}(A - B) = \operatorname{width} A$ and for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A - B)_{i,j} = A_{i,j} - B_{i,j}$.

Let us consider n and let A, B be matrices over $\mathbb{R}$ of dimension n. Then $A \cdot B$ is a matrix over $\mathbb{R}$ of dimension n.

The following propositions are true:

(7) For all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ holds $A + (B - B) = A$.

(8) For all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ holds $(A + B) - B = A$.

(9) For every matrix A over $\mathbb{R}$ holds $(-1) \cdot A = -A$.

(10) For every matrix A over $\mathbb{R}$ and for all elements i, j of $\mathbb{N}$ such that $\langle i, j\rangle \in$ the indices of A holds $(-A)_{i,j} = -A_{i,j}$.

(11) For all real numbers a, b and for every matrix A over $\mathbb{R}$ holds $(a \cdot b) \cdot A = a \cdot (b \cdot A)$.

(12) For all real numbers a, b and for every matrix A over $\mathbb{R}$ holds $(a+b) \cdot A = a \cdot A + b \cdot A$.

(13) For all real numbers a, b and for every matrix A over $\mathbb{R}$ holds $(a-b) \cdot A = a \cdot A - b \cdot A$.

(14) For every matrix A over K such that $n > 0$ and $\operatorname{len} A > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times (\operatorname{len} A)} \cdot A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times (\operatorname{width} A)}$.

(15) For all matrices A, C over K such that $\operatorname{len} A = \operatorname{width} C$ and $\operatorname{len} C > 0$ and $\operatorname{len} A > 0$ holds $(-C) \cdot A = -C \cdot A$.

(16) For all matrices A, B, C over K such that $\operatorname{len} B = \operatorname{len} C$ and $\operatorname{width} B = \operatorname{width} C$ and $\operatorname{len} A = \operatorname{width} B$ and $\operatorname{len} B > 0$ and $\operatorname{len} A > 0$ holds $(B - C) \cdot A = B \cdot A - C \cdot A$.

(17) For all matrices A, B, C over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} C = \operatorname{width} A$ and $\operatorname{len} A > 0$ and $\operatorname{len} C > 0$ holds $(A - B) \cdot C = A \cdot C - B \cdot C$.

(18) For every element m of $\mathbb{N}$ and for all matrices A, C over K such that $\operatorname{width} A > 0$ and $\operatorname{len} A > 0$ holds $A \cdot \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{width} A) \times m} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} A) \times m}$.

(19) For all matrices A, C over K such that $\operatorname{width} A = \operatorname{len} C$ and $\operatorname{len} A > 0$ and $\operatorname{len} C > 0$ holds $A \cdot -C = -A \cdot C$.

(20) For all matrices A, B, C over K such that $\operatorname{len} B = \operatorname{len} C$ and $\operatorname{width} B = \operatorname{width} C$ and $\operatorname{len} B = \operatorname{width} A$ and $\operatorname{len} B > 0$ and $\operatorname{len} A > 0$ holds $A \cdot (B - C) = A \cdot B - A \cdot C$.

(21) For all matrices A, B, C over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{width} C = \operatorname{len} A$ and $\operatorname{len} C > 0$ and $\operatorname{len} A > 0$ holds $C \cdot (A - B) = C \cdot A - C \cdot B$.

(22) Let A, B, C be matrices over $\mathbb{R}$. Suppose that

(i) $\operatorname{len} A = \operatorname{len} B$,

(ii) $\operatorname{width} A = \operatorname{width} B$,

(iii) $\operatorname{len} C = \operatorname{len} A$,

(iv) $\operatorname{width} C = \operatorname{width} A$, and

(v) for all elements i, j of $\mathbb{N}$ such that $\langle i, j \rangle \in$ the indices of A holds $C_{i,j} = A_{i,j} - B_{i,j}$.

Then $C = A - B$.

(23) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_1 > 0$ holds $\operatorname{LineVec2Mx}(x_1 - x_2) = \operatorname{LineVec2Mx} x_1 - \operatorname{LineVec2Mx} x_2$.

(24) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_1 > 0$ holds $\operatorname{ColVec2Mx}(x_1 - x_2) = \operatorname{ColVec2Mx} x_1 - \operatorname{ColVec2Mx} x_2$.

(25) Let A, B be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. Let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} A$, then $\operatorname{Line}(A - B, i) = \operatorname{Line}(A, i) - \operatorname{Line}(B, i)$.

(26) Let A, B be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. Let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{width} A$, then $(A - B)_{\square,i} = A_{\square,i} - B_{\square,i}$.

(27) Let A be a matrix over $\mathbb{R}$ of dimension $n \times k$, B be a matrix over $\mathbb{R}$ of

dimension $k \times m$, and C be a matrix over $\mathbb{R}$ of dimension $m \times l$. If $n > 0$ and $k > 0$ and $m > 0$, then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(28) For all matrices A, B, C over $\mathbb{R}$ of dimension n holds $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(29) For every matrix A over D of dimension n holds $(A^{\mathrm{T}})^{\mathrm{T}} = A$.

(30) For all matrices A, B over $\mathbb{R}$ of dimension n holds $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

(31) For every matrix A over $\mathbb{R}$ such that $n > 0$ and $\operatorname{len} A = n$ and $\operatorname{width} A = m$ holds $-A + A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m}$.

3. Determinants

Let us consider n and let A be a matrix over $\mathbb{R}$ of dimension n. Then $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})A$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n.

Let us consider n and let A be a matrix over $\mathbb{R}$ of dimension n. The functor $\operatorname{Det} A$ yielding a real number is defined as follows:

(Def. 1) $\operatorname{Det} A = \operatorname{Det}(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})A$.

We now state a number of propositions:

(32) For every matrix A over $\mathbb{R}$ of dimension 2 holds $\operatorname{Det} A = A_{1,1} \cdot A_{2,2} - A_{1,2} \cdot A_{2,1}$.

(33) For all finite sequences s_1, s_2, s_3 such that $\operatorname{len} s_1 = n$ and $\operatorname{len} s_2 = n$ and $\operatorname{len} s_3 = n$ holds $\langle s_1, s_2, s_3 \rangle$ is tabular.

(34) Let p_1, p_2, p_3 be finite sequences of elements of D. Suppose $\operatorname{len} p_1 = n$ and $\operatorname{len} p_2 = n$ and $\operatorname{len} p_3 = n$. Then $\langle p_1, p_2, p_3 \rangle$ is a matrix over D of dimension $3 \times n$.

(35) For all elements a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 of D holds $\langle \langle a_1, a_2, a_3 \rangle, \langle b_1, b_2, b_3 \rangle, \langle c_1, c_2, c_3 \rangle \rangle$ is a matrix over D of dimension 3.

(36) Let A be a matrix over D of dimension n, p be a finite sequence of elements of D, and i be a natural number. If $p = A(i)$ and $i \in \operatorname{Seg} n$, then $\operatorname{len} p = n$.

(37) For every matrix A over D of dimension 3 holds $A = \langle \langle A_{1,1}, A_{1,2}, A_{1,3} \rangle, \langle A_{2,1}, A_{2,2}, A_{2,3} \rangle, \langle A_{3,1}, A_{3,2}, A_{3,3} \rangle \rangle$.

(38) Let A be a matrix over $\mathbb{R}$ of dimension 3. Then $\operatorname{Det} A = (((A_{1,1} \cdot A_{2,2} \cdot A_{3,3} - A_{1,3} \cdot A_{2,2} \cdot A_{3,1} - A_{1,1} \cdot A_{2,3} \cdot A_{3,2}) + A_{1,2} \cdot A_{2,3} \cdot A_{3,1}) - A_{1,2} \cdot A_{2,1} \cdot A_{3,3}) + A_{1,3} \cdot A_{2,1} \cdot A_{3,2}$.

(39) For every permutation f of $\operatorname{Seg} 0$ holds $f = \varepsilon_{\mathbb{N}}$.

(40) The permutations of 0-element set $= \{\varepsilon_{\mathbb{N}}\}$.

(41) For every matrix A over K of dimension 0 holds $\operatorname{Det} A = 1_K$.

(42) For every matrix A over $\mathbb{R}$ of dimension 0 holds $\operatorname{Det} A = 1$.

(43) For every natural number n and for every matrix A over K of dimension n holds $\operatorname{Det} A = \operatorname{Det}(A^{\mathrm{T}})$.

(44) For every matrix A over $\mathbb{R}$ of dimension n holds $\operatorname{Det} A = \operatorname{Det}(A^{\mathrm{T}})$.

(45) For all matrices A, B over K of dimension n holds $\operatorname{Det}(A \cdot B) = \operatorname{Det} A \cdot \operatorname{Det} B$.

(46) For all matrices A, B over $\mathbb{R}$ of dimension n holds $\operatorname{Det}(A \cdot B) = \operatorname{Det} A \cdot \operatorname{Det} B$.

4. Matrix as Operator

We now state a number of propositions:

(47) Let x, y be finite sequences of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} A$ and $\operatorname{len} y = \operatorname{len} x$ and $\operatorname{len} x > 0$ and $\operatorname{len} A > 0$, then $(x - y) \cdot A = x \cdot A - y \cdot A$.

(48) Let x, y be finite sequences of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{width} A$ and $\operatorname{len} y = \operatorname{len} x$ and $\operatorname{len} x > 0$ and $\operatorname{len} A > 0$, then $A \cdot (x - y) = A \cdot x - A \cdot y$.

(49) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} A = \operatorname{len} x$ and $\operatorname{len} x > 0$ and $\operatorname{width} A > 0$, then $(-x) \cdot A = -x \cdot A$.

(50) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{width} A$ and $\operatorname{len} A > 0$ and $\operatorname{len} x > 0$, then $A \cdot -x = -A \cdot x$.

(51) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} A$ and $\operatorname{len} x > 0$ and $\operatorname{width} A > 0$, then $x \cdot -A = -x \cdot A$.

(52) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{width} A$ and $\operatorname{len} A > 0$ and $\operatorname{len} x > 0$, then $(-A) \cdot x = -A \cdot x$.

(53) Let a be a real number, x be a finite sequence of elements of $\mathbb{R}$, and A be a matrix over $\mathbb{R}$. If $\operatorname{width} A = \operatorname{len} x$ and $\operatorname{len} x > 0$ and $\operatorname{len} A > 0$, then $A \cdot (a \cdot x) = a \cdot (A \cdot x)$.

(54) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} A$ and $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$, then $x \cdot (A - B) = x \cdot A - x \cdot B$.

(55) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{width} A$ and $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} x > 0$ and $\operatorname{len} A > 0$, then $(A - B) \cdot x = A \cdot x - B \cdot x$.

(56) For every finite sequence x of elements of $\mathbb{R}$ and for every matrix A over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} x$ holds $\operatorname{LineVec2Mx} x \cdot A = \operatorname{LineVec2Mx}(x \cdot A)$.

(57) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} A$ and $\operatorname{width} A = \operatorname{len} B$, then $x \cdot (A \cdot B) = (x \cdot A) \cdot B$.

(58) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{width} A = \operatorname{len} x$ and $\operatorname{len} x > 0$ and $\operatorname{len} A > 0$, then $A \cdot \mathrm{ColVec2Mx}\, x = \mathrm{ColVec2Mx}(A \cdot x)$.

(59) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{width} B$ and $\operatorname{width} A = \operatorname{len} B$ and $\operatorname{len} x > 0$ and $\operatorname{len} B > 0$, then $(A \cdot B) \cdot x = A \cdot (B \cdot x)$.

(60) Let B be a matrix over $\mathbb{R}$ of dimension $n \times m$ and A be a matrix over $\mathbb{R}$ of dimension $m \times k$. Suppose $n > 0$. Let given i, j. If $\langle i, j \rangle \in$ the indices of $B \cdot A$, then $(B \cdot A)_{i,j} = (\mathrm{Line}(B, i) \cdot A)(j)$.

(61) Let A, B be matrices over $\mathbb{R}$ of dimension n and given i, j. If $\langle i, j \rangle \in$ the indices of $B \cdot A$, then $(B \cdot A)_{i,j} = (\mathrm{Line}(B, i) \cdot A)(j)$.

(62) Let A, B be matrices over $\mathbb{R}$ of dimension n. Suppose $n > 0$. Let given i, j. If $\langle i, j \rangle \in$ the indices of $A \cdot B$, then $(A \cdot B)_{i,j} = (A \cdot B_{\square,j})(i)$.

5. Identity and Zero of Matrix of $\mathbb{R}$

Let n be an element of $\mathbb{N}$. The functor $1_{\mathbb{R}}\,\mathrm{matrix}(n)$ yields a matrix over $\mathbb{R}$ of dimension n and is defined as follows:

(Def. 2) $$1_{\mathbb{R}}\,\mathrm{matrix}(n) = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{\mathbb{R}_{\mathrm{F}}}^{n \times n}).$$

One can prove the following propositions:

(63)(i) For every i such that $\langle i, i \rangle \in$ the indices of $1_{\mathbb{R}}\,\mathrm{matrix}(n)$ holds $(1_{\mathbb{R}}\,\mathrm{matrix}(n))_{i,i} = 1$, and

(ii) for all i, j such that $\langle i, j \rangle \in$ the indices of $1_{\mathbb{R}}\,\mathrm{matrix}(n)$ and $i \neq j$ holds $(1_{\mathbb{R}}\,\mathrm{matrix}(n))_{i,j} = 0$.

(64) $(1_{\mathbb{R}}\,\mathrm{matrix}(n))^{\mathrm{T}} = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$.

(65) For all elements n, m of $\mathbb{N}$ such that $n > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m} + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m}$.

(66) For every real number a such that $n > 0$ holds $a \cdot \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m} =$

$$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m}.$$

(67) For every field K and for every matrix A over K holds $A \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{\operatorname{width} A \times \operatorname{width} A} = A$.

(68) For every matrix A over K holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{\operatorname{len} A \times \operatorname{len} A} \cdot A = A$.

(69) For every matrix A over $\mathbb{R}$ holds if $n = \operatorname{width} A$, then $A \cdot 1_{\mathbb{R}} \operatorname{matrix}(n) = A$ and if $m = \operatorname{len} A$, then $1_{\mathbb{R}} \operatorname{matrix}(m) \cdot A = A$.

(70) For every matrix A over $\mathbb{R}$ of dimension n holds $1_{\mathbb{R}} \operatorname{matrix}(n) \cdot A = A$.

(71) For every matrix A over $\mathbb{R}$ of dimension n holds $A \cdot 1_{\mathbb{R}} \operatorname{matrix}(n) = A$.

(72) $\operatorname{Det} 1_{\mathbb{R}} \operatorname{matrix}(n) = 1$.

Let n be an element of $\mathbb{N}$. The functor $0_{\mathbb{R}} \operatorname{matrix}(n)$ yields a matrix over $\mathbb{R}$ of dimension n and is defined by:

(Def. 3) $0_{\mathbb{R}} \operatorname{matrix}(n) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times n}$.

One can prove the following proposition

(73) If $n > 0$, then $\operatorname{Det} 0_{\mathbb{R}} \operatorname{matrix}(n) = 0$.

Let us consider n and let us consider i. The base fin seq(n, i) yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 4) The base fin seq(n, i) $= \operatorname{Replace}(n \mapsto (0$ **qua** element of $\mathbb{R}), i, 1)$.

We now state several propositions:

(74) len (the base fin seq(n, i)) $= n$.

(75) If $1 \le i$ and $i \le n$, then (the base fin seq(n, i))$(i) = 1$.

(76) If $1 \le i$ and $i \le n$ and $1 \le j$ and $j \le n$ and $i \ne j$, then (the base fin seq(n, i))$(j) = 0$.

(77)(i) The base fin seq(1, 1) $= \langle 1 \rangle$,

(ii) the base fin seq(2, 1) $= \langle 1, 0 \rangle$,

(iii) the base fin seq(2, 2) $= \langle 0, 1 \rangle$,

(iv) the base fin seq(3, 1) $= \langle 1, 0, 0 \rangle$,

(v) the base fin seq(3, 2) $= \langle 0, 1, 0 \rangle$, and

(vi) the base fin seq(3, 3) $= \langle 0, 0, 1 \rangle$.

(78) If $1 \le i$ and $i \le n$, then $(1_{\mathbb{R}} \operatorname{matrix}(n))(i) =$ the base fin seq(n, i).

6. Inverse of Matrix

Let n be an element of $\mathbb{N}$ and let A be a matrix over $\mathbb{R}$ of dimension n. We say that A is invertible if and only if:

(Def. 5) There exists a matrix B over $\mathbb{R}$ of dimension n such that $B \cdot A = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$ and $A \cdot B = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$.

Let n be an element of $\mathbb{N}$ and let A be a matrix over $\mathbb{R}$ of dimension n. Let us assume that A is invertible. The functor $\mathrm{Inv}\,A$ yields a matrix over $\mathbb{R}$ of dimension n and is defined as follows:

(Def. 6) $\mathrm{Inv}\,A \cdot A = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$ and $A \cdot \mathrm{Inv}\,A = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$.

Let us consider n. Note that $1_{\mathbb{R}}\,\mathrm{matrix}(n)$ is invertible.

We now state a number of propositions:

(79) $\mathrm{Inv}\,1_{\mathbb{R}}\,\mathrm{matrix}(n) = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$.

(80) For all matrices A, B_1, B_2 over $\mathbb{R}$ of dimension n such that $B_1 \cdot A = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$ and $A \cdot B_2 = 1_{\mathbb{R}}\,\mathrm{matrix}(n)$ holds $B_1 = B_2$ and A is invertible.

(81) For every matrix A over $\mathbb{R}$ of dimension n such that A is invertible holds $\mathrm{Det}\,\mathrm{Inv}\,A = \mathrm{Det}\,A^{-1}$.

(82) For every matrix A over $\mathbb{R}$ of dimension n such that A is invertible holds $\mathrm{Det}\,A \neq 0$.

(83) Let A, B be matrices over $\mathbb{R}$ of dimension n. Suppose A is invertible and B is invertible. Then $A \cdot B$ is invertible and $\mathrm{Inv}\,A \cdot B = \mathrm{Inv}\,B \cdot \mathrm{Inv}\,A$.

(84) For every matrix A over $\mathbb{R}$ of dimension n such that A is invertible holds $\mathrm{Inv}\,\mathrm{Inv}\,A = A$.

(85) $1_{\mathbb{R}}\,\mathrm{matrix}(0) = 0_{\mathbb{R}}\,\mathrm{matrix}(0)$ and $1_{\mathbb{R}}\,\mathrm{matrix}(0) = \emptyset$.

(86) For every finite sequence x of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = n$ and $n > 0$ holds $1_{\mathbb{R}}\,\mathrm{matrix}(n) \cdot x = x$.

(87) Let n be an element of $\mathbb{N}$, x, y be finite sequences of elements of $\mathbb{R}$, and A be a matrix over $\mathbb{R}$ of dimension n. Suppose A is invertible and $\mathrm{len}\,x = n$ and $\mathrm{len}\,y = n$ and $n > 0$. Then $A \cdot x = y$ if and only if $x = \mathrm{Inv}\,A \cdot y$.

(88) For every finite sequence x of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = n$ holds $x \cdot 1_{\mathbb{R}}\,\mathrm{matrix}(n) = x$.

(89) Let x, y be finite sequences of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$ of dimension n. Suppose A is invertible and $\mathrm{len}\,x = n$ and $\mathrm{len}\,y = n$. Then $x \cdot A = y$ if and only if $x = y \cdot \mathrm{Inv}\,A$.

(90) Let A be a matrix over $\mathbb{R}$ of dimension n. Suppose $n > 0$ and A is invertible. Let y be a finite sequence of elements of $\mathbb{R}$. Suppose $\mathrm{len}\,y = n$. Then there exists a finite sequence x of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = n$ and $A \cdot x = y$.

(91) Let A be a matrix over $\mathbb{R}$ of dimension n. Suppose A is invertible. Let y be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} y = n$. Then there exists a finite sequence x of elements of $\mathbb{R}$ such that $\operatorname{len} x = n$ and $x \cdot A = y$.

(92) Let A be a matrix over $\mathbb{R}$ of dimension n and x, y be finite sequences of elements of $\mathbb{R}$. Suppose $\operatorname{len} x = n$ and $\operatorname{len} y = n$ and $x \cdot A = y$. Let j be an element of $\mathbb{N}$. If $1 \leq j$ and $j \leq n$, then $y(j) = |(x, A_{\square,j})|$.

(93) Let A be a matrix over $\mathbb{R}$ of dimension n. Suppose that for every finite sequence y of elements of $\mathbb{R}$ such that $\operatorname{len} y = n$ there exists a finite sequence x of elements of $\mathbb{R}$ such that $\operatorname{len} x = n$ and $x \cdot A = y$. Then there exists a matrix B over $\mathbb{R}$ of dimension n such that $B \cdot A = 1_{\mathbb{R}} \operatorname{matrix}(n)$.

(94) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$ of dimension n. If $n > 0$ and $\operatorname{len} x = n$, then $A^{\mathrm{T}} \cdot x = x \cdot A$.

(95) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$ of dimension n. If $n > 0$ and $\operatorname{len} x = n$, then $x \cdot A^{\mathrm{T}} = A \cdot x$.

(96) Let A be a matrix over $\mathbb{R}$ of dimension n. Suppose that

(i) $n > 0$, and

(ii) for every finite sequence y of elements of $\mathbb{R}$ such that $\operatorname{len} y = n$ there exists a finite sequence x of elements of $\mathbb{R}$ such that $\operatorname{len} x = n$ and $A \cdot x = y$.

Then there exists a matrix B over $\mathbb{R}$ of dimension n such that $A \cdot B = 1_{\mathbb{R}} \operatorname{matrix}(n)$.

(97) Let A be a matrix over $\mathbb{R}$ of dimension n. Suppose that

(i) $n > 0$, and

(ii) for every finite sequence y of elements of $\mathbb{R}$ such that $\operatorname{len} y = n$ there exist finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = n$ and $\operatorname{len} x_2 = n$ and $A \cdot x_1 = y$ and $x_2 \cdot A = y$.

Then A is invertible.

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