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The Rank+Nullity Theorem

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Summary. The rank+nullity theorem states that, if T is a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W, then $\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T)$, where $\mathrm{rank}(T) = \dim(\mathrm{im}(T))$ and $\mathrm{nullity}(T) = \dim(\ker(T))$. The proof treated here is standard; see, for example, [14]: take a basis A of $\ker(T)$ and extend it to a basis B of V, and then show that $\dim(\mathrm{im}(T))$ is equal to $|B - A|$, and that T is one-to-one on $B - A$.

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The articles [21], [11], [32], [22], [19], [33], [34], [7], [2], [17], [10], [18], [8], [9], [20], [1], [12], [3], [5], [6], [27], [29], [24], [31], [25], [13], [4], [30], [28], [26], [23], [15], [16], and [35] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following three propositions:

(1) For all functions f, g such that g is one-to-one and $f \restriction \mathrm{rng}\, g$ is one-to-one and $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$ holds $f \cdot g$ is one-to-one.

(2) For every function f and for all sets X, Y such that $X \subseteq Y$ and $f \restriction Y$ is one-to-one holds $f \restriction X$ is one-to-one.

(3) Let V be a 1-sorted structure and X, Y be subsets of V. Then X meets Y if and only if there exists an element v of V such that $v \in X$ and $v \in Y$.

In the sequel F is a field and V, W are vector spaces over F.

Let F be a field and let V be a finite dimensional vector space over F. One can verify that there exists a basis of V which is finite.

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Let F be a field and let V, W be vector spaces over F. Note that there exists a function from V into W which is linear.

Next we state three propositions:

(4) If Ω_V is finite, then V is finite dimensional.

(5) For every finite dimensional vector space V over F such that $\overline{\overline{\Omega_V}} = 1$ holds $\dim(V) = 0$.

(6) If $\overline{\overline{\Omega_V}} = 2$, then $\dim(V) = 1$.

2. Basic Facts of Linear Transformations

Let F be a field and let V, W be vector spaces over F. A linear transformation from V to W is a linear function from V into W.

In the sequel T is a linear transformation from V to W.

One can prove the following propositions:

(7) For all non empty 1-sorted structures V, W and for every function T from V into W holds $\operatorname{dom} T = \Omega_V$ and $\operatorname{rng} T \subseteq \Omega_W$.

(8) For all elements x, y of V holds $T(x) - T(y) = T(x - y)$.

(9) $T(0_V) = 0_W$.

Let F be a field, let V, W be vector spaces over F, and let T be a linear transformation from V to W. The functor $\ker T$ yielding a strict subspace of V is defined as follows:

(Def. 1) $\Omega_{\ker T} = \{u; u$ ranges over elements of V: $T(u) = 0_W\}$.

We now state the proposition

(10) For every element x of V holds $x \in \ker T$ iff $T(x) = 0_W$.

Let V, W be non empty 1-sorted structures, let T be a function from V into W, and let X be a subset of V. Then $T^\circ X$ is a subset of W.

Let F be a field, let V, W be vector spaces over F, and let T be a linear transformation from V to W. The functor $\operatorname{im} T$ yielding a strict subspace of W is defined as follows:

(Def. 2) $\Omega_{\operatorname{im} T} = T^\circ(\Omega_V)$.

The following propositions are true:

(11) $0_V \in \ker T$.

(12) For every subset X of V holds $T^\circ X$ is a subset of $\operatorname{im} T$.

(13) For every element y of W holds $y \in \operatorname{im} T$ iff there exists an element x of V such that $y = T(x)$.

(14) For every element x of $\ker T$ holds $T(x) = 0_W$.

(15) If T is one-to-one, then $\ker T = \mathbf{0}_V$.

(16) For every finite dimensional vector space V over F holds $\dim(\mathbf{0}_V) = 0$.

(17) For all elements x, y of V such that $T(x) = T(y)$ holds $x - y \in \ker T$.

(18) For every subset A of V and for all elements x, y of V such that $x - y \in \mathrm{Lin}(A)$ holds $x \in \mathrm{Lin}(A \cup \{y\})$.

3. Some Lemmas on Linearly Independent Subsets, Linear Combinations, and Linear Transformations

One can prove the following propositions:

(19) For every subset X of V such that V is a subspace of W holds X is a subset of W.

(20) For every subset A of V such that A is linearly independent holds A is a basis of $\mathrm{Lin}(A)$.

(21) For every subset A of V and for every element x of V such that $x \in \mathrm{Lin}(A)$ and $x \notin A$ holds $A \cup \{x\}$ is linearly dependent.

(22) For every subset A of V and for every basis B of V such that A is a basis of $\ker T$ and $A \subseteq B$ holds $T{\restriction}(B \setminus A)$ is one-to-one.

(23) Let A be a subset of V, l be a linear combination of A, x be an element of V, and a be an element of F. Then $l +\cdot\, (x, a)$ is a linear combination of $A \cup \{x\}$.

Let V be a 1-sorted structure and let X be a subset of V. The functor $V \setminus X$ yields a subset of V and is defined by:

(Def. 3) $V \setminus X = \Omega_V \setminus X$.

Let F be a field, let V be a vector space over F, let l be a linear combination of V, and let X be a subset of V. Then $l^\circ X$ is a subset of F.

In the sequel l is a linear combination of V.

Let F be a field and let V be a vector space over F. Note that there exists a subset of V which is linearly dependent.

Let F be a field, let V be a vector space over F, let l be a linear combination of V, and let A be a subset of V. The functor $l[A]$ yields a linear combination of A and is defined by:

(Def. 4) $l[A] = l{\restriction}A +\cdot (V \setminus A \longmapsto 0_F)$.

The following propositions are true:

(24) $l = l[\text{the support of } l]$.

(25) For every subset A of V and for every element v of V such that $v \in A$ holds $l[A](v) = l(v)$.

(26) For every subset A of V and for every element v of V such that $v \notin A$ holds $l[A](v) = 0_F$.

(27) For all subsets A, B of V and for every linear combination l of B such that $A \subseteq B$ holds $l = l[A] + l[B \setminus A]$.

Let F be a field, let V be a vector space over F, let l be a linear combination of V, and let X be a subset of V. Observe that $l^\circ X$ is finite.

Let V, W be non empty 1-sorted structures, let T be a function from V into W, and let X be a subset of W. Then $T^{-1}(X)$ is a subset of V.

We now state the proposition

(28) For every subset X of V such that X misses the support of l holds $l^\circ X \subseteq \{0_F\}$.

Let F be a field, let V, W be vector spaces over F, let l be a linear combination of V, and let T be a linear transformation from V to W. The functor $T \mathbin{@} l$ yielding a linear combination of W is defined by:

(Def. 5) For every element w of W holds $(T \mathbin{@} l)(w) = \sum(l^\circ T^{-1}(\{w\}))$.

One can prove the following propositions:

(29) $T \mathbin{@} l$ is a linear combination of T°(the support of l).

(30) The support of $T \mathbin{@} l \subseteq T^\circ$(the support of l).

(31) Let l, m be linear combinations of V. Suppose the support of l misses the support of m. Then the support of $l + m =$ (the support of l) $\cup$ (the support of m).

(32) Let l, m be linear combinations of V. Suppose the support of l misses the support of m. Then the support of $l - m =$ (the support of l) $\cup$ (the support of m).

(33) For all subsets A, B of V such that $A \subseteq B$ and B is a basis of V holds V is the direct sum of $\mathrm{Lin}(A)$ and $\mathrm{Lin}(B \setminus A)$.

(34) Let A be a subset of V, l be a linear combination of A, and v be an element of V. Suppose $T{\restriction}A$ is one-to-one and $v \in A$. Then there exists a subset X of V such that X misses A and $T^{-1}(\{T(v)\}) = \{v\} \cup X$.

(35) For every subset X of V such that X misses the support of l and $X \neq \emptyset$ holds $l^\circ X = \{0_F\}$.

(36) For every element w of W such that $w \in$ the support of $T \mathbin{@} l$ holds $T^{-1}(\{w\})$ meets the support of l.

(37) Let v be an element of V. Suppose $T{\restriction}$(the support of l) is one-to-one and $v \in$ the support of l. Then $(T \mathbin{@} l)(T(v)) = l(v)$.

(38) Let G be a finite sequence of elements of V. Suppose $\mathrm{rng}\, G =$ the support of l and $T{\restriction}$(the support of l) is one-to-one. Then $T \cdot (l\, G) = (T \mathbin{@} l)\,(T \cdot G)$.

(39) If $T{\restriction}$(the support of l) is one-to-one, then T°(the support of l) $=$ the support of $T \mathbin{@} l$.

(40) Let A be a subset of V, B be a basis of V, and l be a linear combination of $B \setminus A$. If A is a basis of $\ker T$ and $A \subseteq B$, then $T(\sum l) = \sum(T \mathbin{@} l)$.

(41) Let X be a subset of V. Suppose X is linearly dependent. Then there exists a linear combination l of X such that the support of $l \neq \emptyset$ and

$\sum l = 0_V$.

Let F be a field, let V, W be vector spaces over F, let X be a subset of V, let T be a linear transformation from V to W, and let l be a linear combination of $T^\circ X$. Let us assume that $T \restriction X$ is one-to-one. The functor $T\#l$ yields a linear combination of X and is defined as follows:

(Def. 6) $T\#l = l \cdot T {+\cdot} (V \setminus X \longmapsto 0_F)$.

We now state two propositions:

(42) Let X be a subset of V, l be a linear combination of $T^\circ X$, and v be an element of V. If $v \in X$ and $T \restriction X$ is one-to-one, then $(T\#l)(v) = l(T(v))$.

(43) For every subset X of V and for every linear combination l of $T^\circ X$ such that $T \restriction X$ is one-to-one holds $T \mathbin{@} T\#l = l$.

4. The Rank+Nullity Theorem

Let F be a field, let V, W be finite dimensional vector spaces over F, and let T be a linear transformation from V to W. The functor $\operatorname{rank} T$ yielding a natural number is defined by:

(Def. 7) $\operatorname{rank} T = \dim(\operatorname{im} T)$.

The functor $\operatorname{nullity} T$ yields a natural number and is defined by:

(Def. 8) $\operatorname{nullity} T = \dim(\ker T)$.

Next we state two propositions:

(44) Let V, W be finite dimensional vector spaces over F and T be a linear transformation from V to W. Then $\dim(V) = \operatorname{rank} T + \operatorname{nullity} T$.

(45) Let V, W be finite dimensional vector spaces over F and T be a linear transformation from V to W. If T is one-to-one, then $\dim(V) = \operatorname{rank} T$.

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