

Laplace Expansion

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Summary. In the article the formula for Laplace expansion is proved.

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The notation and terminology used in this paper are introduced in the following articles: [23], [11], [29], [20], [12], [30], [31], [6], [9], [7], [3], [4], [21], [28], [26], [15], [22], [10], [5], [13], [24], [14], [33], [25], [18], [34], [1], [8], [2], [16], [17], [27], [19], and [32].

1. PRELIMINARIES

For simplicity, we follow the rules: x, y are sets, N is an element of \mathbb{N} , c, i, j, k, m, n are natural numbers, D is a non empty set, s is an element of $2\text{Set Seg}(n + 2)$, p is an element of the permutations of n -element set, p_1, q_1 are elements of the permutations of $(n + 1)$ -element set, p_2 is an element of the permutations of $(n + 2)$ -element set, K is a field, a, b are elements of K , f is a finite sequence of elements of K , A is a matrix over K , A_1 is a matrix over D of dimension $n \times m$, p_3 is a finite sequence of elements of D , and M is a matrix over K of dimension n .

The following propositions are true:

- (1) For every finite sequence f and for every natural number i such that $i \in \text{dom } f$ holds $\text{len}(f_{\setminus i}) = \text{len } f - 1$.
- (2) Let i, j, n be natural numbers and M be a matrix over K of dimension n . If $i \in \text{dom } M$, then $\text{len}(\text{the deleting of } i\text{-row and } j\text{-column in } M) = n - 1$.
- (3) If $j \in \text{Seg width } A$, then $\text{width}(\text{the deleting of } j\text{-column in } A) = \text{width } A - 1$.

- (4) For every natural number i such that $\text{len } A > 1$ holds $\text{width } A = \text{width}(\text{the deleting of } i\text{-row in } A)$.
- (5) For every natural number i such that $j \in \text{Seg width } M$ holds $\text{width}(\text{the deleting of } i\text{-row and } j\text{-column in } M) = n - 1$.

Let G be a non empty groupoid, let B be a function from $\{ \text{the carrier of } G, \mathbb{N} \}$ into the carrier of G , let g be an element of G , and let i be a natural number. Then $B(g, i)$ is an element of G .

One can prove the following propositions:

- (6) $\overline{\overline{\text{the permutations of } n\text{-element set}}} = n!$.
- (7) For all i, j such that $i \in \text{Seg}(n + 1)$ and $j \in \text{Seg}(n + 1)$ holds $\overline{\overline{\{p_1 : p_1(i) = j\}}} = n!$.
- (8) Let K be a Fanoian field, given p_2 , and X, Y be elements of $\text{Fin } 2\text{Set Seg}(n + 2)$. Suppose $Y = \{s : s \in X \wedge (\text{Part-sgn}(p_2, K))(s) = -\mathbf{1}_K\}$. Then (the multiplication of K)- $\sum_X \text{Part-sgn}(p_2, K) = \text{power}_K(-\mathbf{1}_K, \text{card } Y)$.
- (9) Let K be a Fanoian field and given p_2, i, j . Suppose $i \in \text{Seg}(n + 2)$ and $p_2(i) = j$. Then there exists an element X of $\text{Fin } 2\text{Set Seg}(n + 2)$ such that $X = \{\{N, i\} : \{N, i\} \in 2\text{Set Seg}(n + 2)\}$ and (the multiplication of K)- $\sum_X \text{Part-sgn}(p_2, K) = \text{power}_K(-\mathbf{1}_K, i + j)$.
- (10) Let given i, j . Suppose $i \in \text{Seg}(n + 1)$ and $j \in \text{Seg}(n + 1)$ and $n \geq 2$. Then there exists a function P_1 from $2\text{Set Seg } n$ into $2\text{Set Seg}(n + 1)$ such that
- (i) $\text{rng } P_1 = 2\text{Set Seg}(n + 1) \setminus \{\{N, i\} : \{N, i\} \in 2\text{Set Seg}(n + 1)\}$,
 - (ii) P_1 is one-to-one, and
 - (iii) for all k, m such that $k < m$ and $\{k, m\} \in 2\text{Set Seg } n$ holds if $m < i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m\}$ and if $m \geq i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m + 1\}$ and if $m \geq i$ and $k \geq i$, then $P_1(\{k, m\}) = \{k + 1, m + 1\}$.
- (11) If $n < 2$, then for every element p of the permutations of n -element set holds p is even and $p = \text{idseq}(n)$.
- (12) Let X, Y, D be non empty sets, f be a function from X into $\text{Fin } Y$, g be a function from $\text{Fin } Y$ into D , and F be a binary operation on D . Suppose that
- (i) for all elements A, B of $\text{Fin } Y$ such that A misses B holds $F(g(A), g(B)) = g(A \cup B)$,
 - (ii) F is commutative and associative and has a unity, and
 - (iii) $g(\emptyset) = \mathbf{1}_F$.

Let I be an element of $\text{Fin } X$. Suppose that for all x, y such that $x \in I$ and $y \in I$ and $f(x)$ meets $f(y)$ holds $x = y$. Then $F\text{-}\sum_I g \cdot f = F\text{-}\sum_{f \circ I} g$ and $F\text{-}\sum_{f \circ I} g = g(\bigcup(f \circ I))$ and $\bigcup(f \circ I)$ is an element of $\text{Fin } Y$.

2. AUXILIARY NOTIONS

Let i, j, n be natural numbers, let us consider K , and let M be a matrix over K of dimension n . Let us assume that $i \in \text{Seg } n$ and $j \in \text{Seg } n$. The functor $\text{Delete}(M, i, j)$ yielding a matrix over K of dimension $n - 1$ is defined as follows:

(Def. 1) $\text{Delete}(M, i, j)$ = the deleting of i -row and j -column in M .

The following propositions are true:

- (13) Let given i, j . Suppose $i \in \text{Seg } n$ and $j \in \text{Seg } n$. Let given k, m such that $k \in \text{Seg}(n - 1)$ and $m \in \text{Seg}(n - 1)$. Then
- (i) if $k < i$ and $m < j$, then $(\text{Delete}(M, i, j))_{k,m} = M_{k,m}$,
 - (ii) if $k < i$ and $m \geq j$, then $(\text{Delete}(M, i, j))_{k,m} = M_{k,m+1}$,
 - (iii) if $k \geq i$ and $m < j$, then $(\text{Delete}(M, i, j))_{k,m} = M_{k+1,m}$, and
 - (iv) if $k \geq i$ and $m \geq j$, then $(\text{Delete}(M, i, j))_{k,m} = M_{k+1,m+1}$.
- (14) For all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ holds $(\text{Delete}(M, i, j))^T = \text{Delete}(M^T, j, i)$.
- (15) For every finite sequence f of elements of K and for all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ holds $\text{Delete}(M, i, j) = \text{Delete}(\text{RLine}(M, i, f), i, j)$.

Let us consider c, n, m, D , let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D . The functor $\text{ReplaceCol}(M, c, p_3)$ yielding a matrix over D of dimension $n \times m$ is defined by:

- (Def. 2)(i) $\text{len } \text{ReplaceCol}(M, c, p_3) = \text{len } M$ and $\text{width } \text{ReplaceCol}(M, c, p_3) = \text{width } M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $j \neq c$, then $(\text{ReplaceCol}(M, c, p_3))_{i,j} = M_{i,j}$ and if $j = c$, then $(\text{ReplaceCol}(M, c, p_3))_{i,c} = p_3(i)$ if $\text{len } p_3 = \text{len } M$,
- (ii) $\text{ReplaceCol}(M, c, p_3) = M$, otherwise.

Let us consider c, n, m, D , let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D . We introduce $\text{RCol}(M, c, p_3)$ as a synonym of $\text{ReplaceCol}(M, c, p_3)$.

We now state four propositions:

- (16) For every i such that $i \in \text{Seg } \text{width } A_1$ holds if $i = c$ and $\text{len } p_3 = \text{len } A_1$, then $(\text{RCol}(A_1, c, p_3))_{\square, i} = p_3$ and if $i \neq c$, then $(\text{RCol}(A_1, c, p_3))_{\square, i} = (A_1)_{\square, i}$.
- (17) If $c \notin \text{Seg } \text{width } A_1$, then $\text{RCol}(A_1, c, p_3) = A_1$.
- (18) $\text{RCol}(A_1, c, (A_1)_{\square, c}) = A_1$.
- (19) Let A be a matrix over D of dimension $n \times m$ and A' be a matrix over D of dimension $m \times n$. If $A' = A^T$ and if $m = 0$, then $n = 0$, then $\text{ReplaceCol}(A, c, p_3) = (\text{ReplaceLine}(A', c, p_3))^T$.

3. PERMUTATIONS

Let us consider i, n and let p_4 be an element of the permutations of $(n+1)$ -element set. Let us assume that $i \in \text{Seg}(n+1)$. The functor $\text{Rem}(p_4, i)$ yielding an element of the permutations of n -element set is defined by the condition (Def. 3).

- (Def. 3) Let given k such that $k \in \text{Seg } n$. Then
- (i) if $k < i$, then if $p_4(k) < p_4(i)$, then $(\text{Rem}(p_4, i))(k) = p_4(k)$ and if $p_4(k) \geq p_4(i)$, then $(\text{Rem}(p_4, i))(k) = p_4(k) - 1$, and
 - (ii) if $k \geq i$, then if $p_4(k+1) < p_4(i)$, then $(\text{Rem}(p_4, i))(k) = p_4(k+1)$ and if $p_4(k+1) \geq p_4(i)$, then $(\text{Rem}(p_4, i))(k) = p_4(k+1) - 1$.

One can prove the following three propositions:

- (20) Let given i, j . Suppose $i \in \text{Seg}(n+1)$ and $j \in \text{Seg}(n+1)$. Let P be a set. Suppose $P = \{p_1 : p_1(i) = j\}$. Then there exists a function P_1 from P into the permutations of n -element set such that P_1 is bijective and for every q_1 such that $q_1(i) = j$ holds $P_1(q_1) = \text{Rem}(q_1, i)$.
- (21) For all i, j such that $i \in \text{Seg}(n+1)$ and $p_1(i) = j$ holds $(-1)^{\text{sgn}(p_1)} a = \text{power}_K(-\mathbf{1}_K, i+j) \cdot (-1)^{\text{sgn}(\text{Rem}(p_1, i))} a$.
- (22) Let given i, j . Suppose $i \in \text{Seg}(n+1)$ and $p_1(i) = j$. Let M be a matrix over K of dimension $n+1$ and D_1 be a matrix over K of dimension n . Suppose $D_1 = \text{Delete}(M, i, j)$. Then (the product on paths of M)(p_1) = $\text{power}_K(-\mathbf{1}_K, i+j) \cdot M_{i,j} \cdot (\text{the product on paths of } D_1)(\text{Rem}(p_1, i))$.

4. MINORS AND COFACTORS

Let i, j, n be natural numbers, let us consider K , and let M be a matrix over K of dimension n . The functor $\text{Minor}(M, i, j)$ yielding an element of K is defined by:

- (Def. 4) $\text{Minor}(M, i, j) = \text{Det Delete}(M, i, j)$.

Let i, j, n be natural numbers, let us consider K , and let M be a matrix over K of dimension n . The functor $\text{Cofactor}(M, i, j)$ yielding an element of K is defined as follows:

- (Def. 5) $\text{Cofactor}(M, i, j) = \text{power}_K(-\mathbf{1}_K, i+j) \cdot \text{Minor}(M, i, j)$.

The following propositions are true:

- (23) Let given i, j . Suppose $i \in \text{Seg } n$ and $j \in \text{Seg } n$. Let P be an element of Fin (the permutations of n -element set). Suppose $P = \{p : p(i) = j\}$. Let M be a matrix over K of dimension n . Then (the addition of K)- \sum_P (the product on paths of M) = $M_{i,j} \cdot \text{Cofactor}(M, i, j)$.
- (24) For all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ holds $\text{Minor}(M, i, j) = \text{Minor}(M^T, j, i)$.

Let us consider n, K and let M be a matrix over K of dimension n . The matrix of cofactor M yielding a matrix over K of dimension n is defined by the condition (Def. 6).

- (Def. 6) Let i, j be natural numbers. Suppose $\langle i, j \rangle \in$ the indices of the matrix of cofactor M . Then (the matrix of cofactor M) $_{i,j} = \text{Cofactor}(M, i, j)$.

5. LAPLACE EXPANSION

Let us consider n, i, K and let M be a matrix over K of dimension n . The functor $\text{LaplaceExpL}(M, i)$ yields a finite sequence of elements of K and is defined as follows:

- (Def. 7) $\text{len LaplaceExpL}(M, i) = n$ and for every j such that $j \in \text{dom LaplaceExpL}(M, i)$ holds
 $(\text{LaplaceExpL}(M, i))(j) = M_{i,j} \cdot \text{Cofactor}(M, i, j)$.

Let us consider n , let j be a natural number, let us consider K , and let M be a matrix over K of dimension n . The functor $\text{LaplaceExpC}(M, j)$ yields a finite sequence of elements of K and is defined by:

- (Def. 8) $\text{len LaplaceExpC}(M, j) = n$ and for every i such that $i \in \text{dom LaplaceExpC}(M, j)$ holds $(\text{LaplaceExpC}(M, j))(i) = M_{i,j} \cdot \text{Cofactor}(M, i, j)$.

One can prove the following propositions:

- (25) For every natural number i and for every matrix M over K of dimension n such that $i \in \text{Seg } n$ holds $\text{Det } M = \sum \text{LaplaceExpL}(M, i)$.
- (26) For every i such that $i \in \text{Seg } n$ holds $\text{LaplaceExpC}(M, i) = \text{LaplaceExpL}(M^T, i)$.
- (27) For every natural number j and for every matrix M over K of dimension n such that $j \in \text{Seg } n$ holds $\text{Det } M = \sum \text{LaplaceExpC}(M, j)$.
- (28) For all p, i such that $\text{len } f = n$ and $i \in \text{Seg } n$ holds $\text{Line}(\text{the matrix of cofactor } M, i) \bullet f = \text{LaplaceExpL}(\text{RLine}(M, i, f), i)$.
- (29) If $i \in \text{Seg } n$, then $\text{Line}(M, j) \cdot ((\text{the matrix of cofactor } M)^T)_{\square, i} = \text{Det RLine}(M, i, \text{Line}(M, j))$.
- (30) If $\text{Det } M \neq 0_K$, then $M \cdot (\text{Det } M^{-1} \cdot (\text{the matrix of cofactor } M)^T) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n, K}$.
- (31) For all f, i such that $\text{len } f = n$ and $i \in \text{Seg } n$ holds $(\text{the matrix of cofactor } M)_{\square, i} \bullet f = \text{LaplaceExpL}(\text{RLine}(M^T, i, f), i)$.
- (32) If $i \in \text{Seg } n$ and $j \in \text{Seg } n$, then $\text{Line}((\text{the matrix of cofactor } M)^T, i) \cdot M_{\square, j} = \text{Det RLine}(M^T, i, \text{Line}(M^T, j))$.

- (33) If $\text{Det } M \neq 0_K$, then $\text{Det } M^{-1} \cdot (\text{the matrix of cofactor } M)^T \cdot M =$

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}_{n \times n}^K.$$
- (34) M is invertible iff $\text{Det } M \neq 0_K$.
- (35) If $\text{Det } M \neq 0_K$, then $M^\smile = \text{Det } M^{-1} \cdot (\text{the matrix of cofactor } M)^T$.
- (36) Let M be a matrix over K of dimension n . Suppose M is invertible. Let given i, j . If $\langle i, j \rangle \in$ the indices of M^\smile , then $M^\smile_{i,j} = \text{Det } M^{-1} \cdot \text{power}_K(-\mathbf{1}_K, i + j) \cdot \text{Minor}(M, j, i)$.
- (37) Let A be a matrix over K of dimension n . Suppose $\text{Det } A \neq 0_K$. Let x, b be matrices over K . Suppose $\text{len } x = n$ and $A \cdot x = b$. Then $x = A^\smile \cdot b$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \text{Det } A^{-1} \cdot \text{Det ReplaceCol}(A, i, b_{\square, j})$.
- (38) Let A be a matrix over K of dimension n . Suppose $\text{Det } A \neq 0_K$. Let x, b be matrices over K . Suppose $\text{width } x = n$ and $x \cdot A = b$. Then $x = b \cdot A^\smile$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \text{Det } A^{-1} \cdot \text{Det ReplaceLine}(A, j, \text{Line}(b, i))$.

6. PRODUCT BY A VECTOR

Let D be a non empty set and let f be a finite sequence of elements of D . Then $\langle f \rangle$ is a matrix over D of dimension $1 \times \text{len } f$.

Let us consider K , let M be a matrix over K , and let f be a finite sequence of elements of K . The functor $M \cdot f$ yielding a matrix over K is defined by:

$$\text{(Def. 9)} \quad M \cdot f = M \cdot \langle f \rangle^T.$$

The functor $f \cdot M$ yields a matrix over K and is defined by:

$$\text{(Def. 10)} \quad f \cdot M = \langle f \rangle \cdot M.$$

Next we state two propositions:

- (39) Let A be a matrix over K of dimension n . Suppose $\text{Det } A \neq 0_K$. Let x, b be finite sequences of elements of K . Suppose $\text{len } x = n$ and $A \cdot x = \langle b \rangle^T$. Then $\langle x \rangle^T = A^\smile \cdot b$ and for every i such that $i \in \text{Seg } n$ holds $x(i) = \text{Det } A^{-1} \cdot \text{Det ReplaceCol}(A, i, b)$.
- (40) Let A be a matrix over K of dimension n . Suppose $\text{Det } A \neq 0_K$. Let x, b be finite sequences of elements of K . Suppose $\text{len } x = n$ and $x \cdot A = \langle b \rangle$. Then $\langle x \rangle = b \cdot A^\smile$ and for every i such that $i \in \text{Seg } n$ holds $x(i) = \text{Det } A^{-1} \cdot \text{Det ReplaceLine}(A, i, b)$.

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