

Arrow's Impossibility Theorem

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Summary. A formalization of the first proof from [6].

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The terminology and notation used here are introduced in the following articles: [11], [13], [12], [10], [9], [5], [2], [3], [1], [8], [4], and [7].

1. PRELIMINARIES

Let A , B' be non empty sets, let B be a non empty subset of B' , let f be a function from A into B , and let x be an element of A . Then $f(x)$ is an element of B .

Next we state two propositions:

- (1) For every finite set A such that $\text{card } A \geq 2$ and for every element a of A there exists an element b of A such that $b \neq a$.
- (2) Let A be a finite set. Suppose $\text{card } A \geq 3$. Let a, b be elements of A . Then there exists an element c of A such that $c \neq a$ and $c \neq b$.

2. LINEAR PREORDERS AND LINEAR ORDERS

In the sequel A denotes a non empty set and a, b, c denote elements of A .

Let us consider A . The functor $\text{LinPreorders } A$ is defined by the condition (Def. 1).

(Def. 1) Let R be a set. Then $R \in \text{LinPreorders } A$ if and only if the following conditions are satisfied:

- (i) R is a binary relation on A ,
- (ii) for all a, b holds $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$, and
- (iii) for all a, b, c such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$ holds $\langle a, c \rangle \in R$.

Let us consider A . Note that $\text{LinPreorders } A$ is non empty.

Let us consider A . The functor $\text{LinOrders } A$ yielding a subset of $\text{LinPreorders } A$ is defined by:

(Def. 2) For every element R of $\text{LinPreorders } A$ holds $R \in \text{LinOrders } A$ iff for all a, b such that $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$ holds $a = b$.

Let A be a set. One can verify that there exists an order in A which is connected.

Let us consider A . Then $\text{LinOrders } A$ can be characterized by the condition:

(Def. 3) For every set R holds $R \in \text{LinOrders } A$ iff R is a connected order in A .

Let us consider A . One can verify that $\text{LinOrders } A$ is non empty.

In the sequel o, o' are elements of $\text{LinPreorders } A$ and o'' is an element of $\text{LinOrders } A$.

Let us consider A, o, a, b . The predicate $a \leq_o b$ is defined by:

(Def. 4) $\langle a, b \rangle \in o$.

Let us consider A, o, a, b . We introduce $b \geq_o a$ as a synonym of $a \leq_o b$. We introduce $b <_o a$ as an antonym of $a \leq_o b$. We introduce $a >_o b$ as an antonym of $a \leq_o b$.

We now state a number of propositions:

- (3) $a \leq_o a$.
- (4) $a \leq_o b$ or $b \leq_o a$.
- (5) If $a \leq_o b$ or $a <_o b$ and if $b \leq_o c$ or $b <_o c$, then $a \leq_o c$.
- (6) If $a \leq_{o'} b$ and $b \leq_{o''} a$, then $a = b$.
- (7) If $a \neq b$ and $b \neq c$ and $a \neq c$, then there exists o such that $a <_o b$ and $b <_o c$.
- (8) There exists o such that for every a such that $a \neq b$ holds $b <_o a$.
- (9) There exists o such that for every a such that $a \neq b$ holds $a <_o b$.
- (10) If $a \neq b$ and $a \neq c$, then there exists o such that $a <_o b$ and $a <_o c$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.
- (11) If $a \neq b$ and $a \neq c$, then there exists o such that $b <_o a$ and $c <_o a$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.
- (12) Let o, o' be elements of $\text{LinOrders } A$. Then $a <_o b$ iff $a <_{o'} b$ and $b <_o a$ iff $b <_{o'} a$ if and only if $a <_o b$ iff $a <_{o'} b$.
- (13) Let o be an element of $\text{LinOrders } A$ and o' be an element of $\text{LinPreorders } A$. Then for all a, b such that $a <_o b$ holds $a <_{o'} b$ if and only

if for all a, b holds $a <_o b$ iff $a <_{o'} b$.

3. ARROW'S THEOREM

For simplicity, we follow the rules: A, N are finite non empty sets, a, b are elements of A , i, n are elements of N , p, p' are elements of $(\text{LinPreorders } A)^N$, and f is a function from $(\text{LinPreorders } A)^N$ into $\text{LinPreorders } A$.

We now state the proposition

(14) Suppose that

- (i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,
- (ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ and $b <_{p(i)} a$ iff $b <_{p'(i)} a$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and
- (iii) $\text{card } A \geq 3$.

Then there exists n such that for all p, a, b such that $a <_{p(n)} b$ holds $a <_{f(p)} b$.

In the sequel p, p' denote elements of $(\text{LinOrders } A)^N$ and f denotes a function from $(\text{LinOrders } A)^N$ into $\text{LinPreorders } A$.

One can prove the following proposition

(15) Suppose that

- (i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,
- (ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and
- (iii) $\text{card } A \geq 3$.

Then there exists n such that for all p, a, b holds $a <_{p(n)} b$ iff $a <_{f(p)} b$.

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [5] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [6] John Geanakoplos. Three brief proofs of arrow's impossibility theorem. Cowles Foundation Discussion Papers 1123R3, Cowles Foundation, Yale University, April 1996. Available at <http://ideas.repec.org/p/cwl/cwldpp/1123r3.html>.
- [7] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [8] Andrzej Trybulec. Function domains and Fränkel operator. *Formalized Mathematics*, 1(3):495–500, 1990.
- [9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [10] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [11] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [12] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

- [13] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(1):85–89, 1990.

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