

Several Integrability Formulas of Special Functions

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Summary. In this article, we give several integrability formulas of special and composite functions including trigonometric function, inverse trigonometric function, hyperbolic function and logarithmic function.

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The notation and terminology used here are introduced in the following papers: [21], [20], [7], [12], [6], [25], [3], [8], [26], [24], [5], [22], [18], [19], [17], [9], [16], [11], [14], [1], [15], [23], [13], [10], [2], and [4].

1. PRELIMINARIES

For simplicity, we adopt the following convention: f, f_1, f_2, g denote partial functions from \mathbb{R} to \mathbb{R} , A denotes a closed-interval subset of \mathbb{R} , r, x, x_0 denote real numbers, n denotes an element of \mathbb{N} , and Z denotes an open subset of \mathbb{R} .

The following propositions are true:

- (1) $\sin(x + 2 \cdot n \cdot \pi) = \sin x$.
- (2) $\sin(x + (2 \cdot n + 1) \cdot \pi) = -\sin x$.
- (3) $\cos(x + 2 \cdot n \cdot \pi) = \cos x$.

- (4) $\cos(x + (2 \cdot n + 1) \cdot \pi) = -\cos x$.
- (5) If $\sin(\frac{x}{2}) \geq 0$, then $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos x}{2}}$.
- (6) If $\sin(\frac{x}{2}) < 0$, then $\sin(\frac{x}{2}) = -\sqrt{\frac{1-\cos x}{2}}$.
- (7) $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.
- (8) $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$.
- (9) $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \subseteq]-1, 1[$.
- (10) $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.
- (11) $\arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$.
- (12) If $\cos(\frac{x}{2}) \geq 0$, then $\cos(\frac{x}{2}) = \sqrt{\frac{1+\cos x}{2}}$.
- (13) $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.
- (14) $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$.
- (15) $\arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.
- (16) $\arccos(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$.
- (17) (The function \sinh)(1) = $\frac{e^2-1}{2 \cdot e}$.
- (18) (The function \cosh)(0) = 1.
- (19) (The function \cosh)(1) = $\frac{e^2+1}{2 \cdot e}$.
- (20) For every linear function L_1 holds $-L_1$ is a linear function.
- (21) For every rest R_1 holds $-R_1$ is a rest.
- (22) For all f_1, x_0 such that f_1 is differentiable in x_0 holds $-f_1$ is differentiable in x_0 and $(-f_1)'(x_0) = -f_1'(x_0)$.
- (23) Let given f_1, Z . Suppose $Z \subseteq \text{dom}(-f_1)$ and f_1 is differentiable on Z . Then $-f_1$ is differentiable on Z and for every x such that $x \in Z$ holds $(-f_1)'|_Z(x) = -f_1'(x)$.
- (24) $-$ the function \sin is differentiable on \mathbb{R} .
- (25) $-$ the function \cos is differentiable in x and $(-the\ function\ \cos)'(x) = (the\ function\ \sin)(x)$.
- (26)(i) $-$ the function \cos is differentiable on \mathbb{R} , and
(ii) for every x such that $x \in \mathbb{R}$ holds $(-the\ function\ \cos)'(x) = (the\ function\ \sin)(x)$.
- (27) (The function \sin)'| $_{\mathbb{R}}$ = the function \cos .
- (28) (The function \cos)'| $_{\mathbb{R}}$ = $-$ the function \sin .
- (29) ($-$ the function \cos)'| $_{\mathbb{R}}$ = the function \sin .
- (30) (The function \sinh)'| $_{\mathbb{R}}$ = the function \cosh .
- (31) (The function \cosh)'| $_{\mathbb{R}}$ = the function \sinh .
- (32) (The function \exp)'| $_{\mathbb{R}}$ = the function \exp .

- (33) Suppose $Z \subseteq \text{dom}(\text{the function tan})$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^2}$ and $(\text{the function cos})(x) \neq 0$. Then
- (i) the function tan is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{the function tan})'_{|Z}(x) = \frac{1}{(\text{the function cos})(x)^2}$.
- (34) Suppose that
- (i) $Z \subseteq \text{dom}(\text{the function cot})$, and
 - (ii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\text{the function sin})(x)^2}$ and $(\text{the function sin})(x) \neq 0$.
- Then
- (iii) the function cot is differentiable on Z , and
 - (iv) for every x such that $x \in Z$ holds $(\text{the function cot})'_{|Z}(x) = -\frac{1}{(\text{the function sin})(x)^2}$.
- (35) For every real number r holds $\text{dom}(\mathbb{R} \mapsto r) = \mathbb{R}$ and $\text{rng}(\mathbb{R} \mapsto r) \subseteq \mathbb{R}$.

Let r be a real number. The functor $\text{Cst } r$ yielding a function from \mathbb{R} into \mathbb{R} is defined as follows:

(Def. 1) $\text{Cst } r = \mathbb{R} \mapsto r$.

We now state two propositions:

- (36) For all real numbers a, b and for every closed-interval subset A of \mathbb{R} holds $\chi_{A,A} = \text{Cst } 1|_A$.
- (37) For all real numbers a, b and for every closed-interval subset A of \mathbb{R} such that $A = [a, b]$ holds $\sup A = b$ and $\inf A = a$.

2. SEVERAL INTEGRABILITY FORMULAS OF SPECIAL FUNCTIONS

The following propositions are true:

- (38) For all real numbers a, b such that $a \leq b$ holds $\int_a^b \text{Cst } 1(x)dx = b - a$.
- (39) $\int_A (\text{the function cos})(x)dx = (\text{the function sin})(\sup A) - (\text{the function sin})(\inf A)$.
- (40) If $A = [0, \frac{\pi}{2}]$, then $\int_A (\text{the function cos})(x)dx = 1$.
- (41) If $A = [0, \pi]$, then $\int_A (\text{the function cos})(x)dx = 0$.
- (42) If $A = [0, \frac{\pi-3}{2}]$, then $\int_A (\text{the function cos})(x)dx = -1$.

- (43) If $A = [0, \pi \cdot 2]$, then $\int_A (\text{the function } \cos)(x)dx = 0$.
- (44) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A (\text{the function } \cos)(x)dx = 0$.
- (45) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A (\text{the function } \cos)(x)dx = -2 \cdot \sin x$.
- (46) $\int_A (-\text{the function } \sin)(x)dx = (\text{the function } \cos)(\sup A) - (\text{the function } \cos)(\inf A)$.
- (47) If $A = [0, \frac{\pi}{2}]$, then $\int_A (-\text{the function } \sin)(x)dx = -1$.
- (48) If $A = [0, \pi]$, then $\int_A (-\text{the function } \sin)(x)dx = -2$.
- (49) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int_A (-\text{the function } \sin)(x)dx = -1$.
- (50) If $A = [0, \pi \cdot 2]$, then $\int_A (-\text{the function } \sin)(x)dx = 0$.
- (51) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A (-\text{the function } \sin)(x)dx = -2$.
- (52) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A (-\text{the function } \sin)(x)dx = -2 \cdot \cos x$.
- (53) $\int_A (\text{the function } \exp)(x)dx = (\text{the function } \exp)(\sup A) - (\text{the function } \exp)(\inf A)$.
- (54) If $A = [0, 1]$, then $\int_A (\text{the function } \exp)(x)dx = e - 1$.
- (55) $\int_A (\text{the function } \sinh)(x)dx = (\text{the function } \cosh)(\sup A) - (\text{the function } \cosh)(\inf A)$.
- (56) If $A = [0, 1]$, then $\int_A (\text{the function } \sinh)(x)dx = \frac{(e - 1)^2}{2 \cdot e}$.
- (57) $\int_A (\text{the function } \cosh)(x)dx = (\text{the function } \sinh)(\sup A) - (\text{the function } \sinh)(\inf A)$.

$\sinh)(\inf A)$.

(58) If $A = [0, 1]$, then $\int_A (\text{the function cosh})(x)dx = \frac{e^2 - 1}{2 \cdot e}$.

(59) Suppose that

- (i) $A \subseteq Z$,
- (ii) $\text{dom}(\text{the function tan}) = Z$,
- (iii) $\text{dom}(\text{the function tan}) = \text{dom } f_2$,
- (iv) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(\text{the function cos})(x)^2}$ and (the function $\cos)(x) \neq 0$, and
- (v) f_2 is continuous on A .

Then $\int_A f_2(x)dx = (\text{the function tan})(\sup A) - (\text{the function tan})(\inf A)$.

(60) Suppose that

- (i) $A \subseteq Z$,
- (ii) $\text{dom}(\text{the function cot}) = Z$,
- (iii) $\text{dom}(\text{the function cot}) = \text{dom } f_2$,
- (iv) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(\text{the function sin})(x)^2}$ and (the function $\sin)(x) \neq 0$, and
- (v) f_2 is continuous on A .

Then $\int_A f_2(x)dx = (\text{the function cot})(\sup A) - (\text{the function cot})(\inf A)$.

(61) Suppose $\text{dom}(\text{the function tanh}) = \text{dom } f_2$ and for every x such that $x \in \mathbb{R}$ holds $f_2(x) = \frac{1}{(\text{the function cosh})(x)^2}$ and f_2 is continuous on A . Then

$\int_A f_2(x)dx = (\text{the function tanh})(\sup A) - (\text{the function tanh})(\inf A)$.

(62) Suppose $A \subseteq]-1, 1[$ and $\text{dom}((\text{the function arcsin})'_{] -1, 1[}) = \text{dom } f_2$ and for every x holds $x \in]-1, 1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^2}}$ and f_2 is continuous on A . Then $\int_A f_2(x)dx = (\text{the function arcsin})(\sup A) - (\text{the function arcsin})(\inf A)$.

(63) Suppose $A \subseteq]-1, 1[$ and $\text{dom}((\text{the function arccos})'_{] -1, 1[}) = \text{dom } f_2$ and for every x holds $x \in]-1, 1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^2}}$ and f_2 is continuous on A . Then $\int_A f_2(x)dx = (\text{the function arccos})(\sup A) - (\text{the function arccos})(\inf A)$.

(64) Suppose that

- (i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,
- (ii) $\text{dom}((\text{the function arcsin})'_{] -1, 1[}) = \text{dom } f_2$,

(iii) for every x holds $x \in]-1, 1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^2}}$, and

(iv) f_2 is continuous on A .

$$\text{Then } \int_A f_2(x) dx = \frac{\pi}{2}.$$

(65) Suppose that

(i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,

(ii) $\text{dom}((\text{the function } \arccos)'_{] -1, 1[}) = \text{dom } f_2$,

(iii) for every x holds $x \in]-1, 1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^2}}$, and

(iv) f_2 is continuous on A .

$$\text{Then } \int_A f_2(x) dx = -\frac{\pi}{2}.$$

(66) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{|Z}$ is integrable on A and $f'_{|Z}$ is bounded on A and $g'_{|Z}$ is

$$\text{integrable on } A \text{ and } g'_{|Z} \text{ is bounded on } A. \text{ Then } \int_A (f'_{|Z} + g'_{|Z})(x) dx = \\ ((f(\sup A) - f(\inf A)) + g(\sup A)) - g(\inf A).$$

(67) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{|Z}$ is integrable on A and $f'_{|Z}$ is bounded on A and $g'_{|Z}$ is

$$\text{integrable on } A \text{ and } g'_{|Z} \text{ is bounded on } A. \text{ Then } \int_A (f'_{|Z} - g'_{|Z})(x) dx = \\ f(\sup A) - f(\inf A) - (g(\sup A) - g(\inf A)).$$

(68) Suppose f is differentiable on Z and $A \subseteq Z$ and $f'_{|Z}$ is integrable on A and $f'_{|Z}$ is bounded on A . Then $\int_A (r f'_{|Z})(x) dx = r \cdot f(\sup A) - r \cdot f(\inf A)$.

(69) $\int_A ((\text{the function } \sin) + (\text{the function } \cos))(x) dx = (((-\text{the function } \cos) \\ (\sup A) - (-\text{the function } \cos)(\inf A)) + (\text{the function } \sin)(\sup A)) - (\text{the} \\ \text{function } \sin)(\inf A).$

(70) If $A = [0, \frac{\pi}{2}]$, then $\int_A ((\text{the function } \sin) + (\text{the function } \cos))(x) dx = 2$.

(71) If $A = [0, \pi]$, then $\int_A ((\text{the function } \sin) + (\text{the function } \cos))(x) dx = 2$.

(72) If $A = [0, \frac{\pi-3}{2}]$, then $\int_A ((\text{the function } \sin) + (\text{the function } \cos))(x) dx = 0$.

(73) If $A = [0, \pi \cdot 2]$, then $\int_A ((\text{the function } \sin) + (\text{the function } \cos))(x) dx =$

- 0.
- (74) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then

$$\int_A ((\text{the function sin}) + (\text{the function cos}))(x) dx = 2.$$
- (75) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then

$$\int_A ((\text{the function sin}) + (\text{the function cos}))(x) dx = 2 \cdot \cos x - 2 \cdot \sin x.$$
- (76)
$$\int_A ((\text{the function sinh}) + (\text{the function cosh}))(x) dx = ((\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A)) + (\text{the function sinh})(\sup A) - (\text{the function sinh})(\inf A).$$
- (77) If $A = [0, 1]$, then
$$\int_A ((\text{the function sinh}) + (\text{the function cosh}))(x) dx = e - 1.$$
- (78)
$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = (-\text{the function cos})(\sup A) - (-\text{the function cos})(\inf A) - ((\text{the function sin})(\sup A) - (\text{the function sin})(\inf A)).$$
- (79) If $A = [0, \frac{\pi}{2}]$, then
$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 0.$$
- (80) If $A = [0, \pi]$, then
$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 2.$$
- (81) If $A = [0, \frac{\pi \cdot 3}{2}]$, then
$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 2.$$
- (82) If $A = [0, \pi \cdot 2]$, then
$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 0.$$
- (83) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then

$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 2.$$
- (84) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then

$$\int_A ((\text{the function sin}) - (\text{the function cos}))(x) dx = 2 \cdot \cos x + 2 \cdot \sin x.$$
- (85)
$$\int_A (r (\text{the function sin}))(x) dx = r \cdot (-\text{the function cos})(\sup A) - r \cdot (-\text{the function cos})(\inf A).$$

$$(86) \quad \int_A (r(\text{the function cos}))(x)dx = r \cdot (\text{the function sin})(\sup A) - r \cdot (\text{the function sin})(\inf A).$$

$$(87) \quad \int_A (r(\text{the function sinh}))(x)dx = r \cdot (\text{the function cosh})(\sup A) - r \cdot (\text{the function cosh})(\inf A).$$

$$(88) \quad \int_A (r(\text{the function cosh}))(x)dx = r \cdot (\text{the function sinh})(\sup A) - r \cdot (\text{the function sinh})(\inf A).$$

$$(89) \quad \int_A (r(\text{the function exp}))(x)dx = r \cdot (\text{the function exp})(\sup A) - r \cdot (\text{the function exp})(\inf A).$$

$$(90) \quad \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = \frac{1}{2} \cdot ((\text{the function cos})(\inf A) \cdot (\text{the function cos})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function cos})(\sup A)).$$

$$(91) \quad \text{If } A = [0, \frac{\pi}{2}], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = \frac{1}{2}.$$

$$(92) \quad \text{If } A = [0, \pi], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = 0.$$

$$(93) \quad \text{If } A = [0, \pi \cdot \frac{3}{2}], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = \frac{1}{2}.$$

$$(94) \quad \text{If } A = [0, \pi \cdot 2], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = 0.$$

$$(95) \quad \text{If } A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = 0.$$

$$(96) \quad \text{If } A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi], \text{ then } \int_A ((\text{the function sin})(\text{the function cos}))(x)dx = 0.$$

$$(97) \quad \int_A ((\text{the function sin})(\text{the function sin}))(x)dx = ((\text{the function cos})(\inf A) \cdot (\text{the function sin})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function sin})(\sup A)) + \int_A ((\text{the function cos})(\text{the function cos}))(x)dx.$$

- (98) $\int_A ((\text{the function sinh}) (\text{the function sinh}))(x)dx = (\text{the function cosh})$
 $(\sup A) \cdot (\text{the function sinh})(\sup A) - (\text{the function cosh})(\inf A) \cdot (\text{the func-}$
 $\text{tion sinh})(\inf A) - \int_A ((\text{the function cosh}) (\text{the function cosh}))(x)dx.$
- (99) $\int_A ((\text{the function sinh}) (\text{the function cosh}))(x)dx = \frac{1}{2} \cdot ((\text{the function}$
 $\text{cosh})(\sup A) \cdot (\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A) \cdot (\text{the}$
 $\text{function cosh})(\inf A)).$
- (100) $\int_A ((\text{the function exp}) (\text{the function exp}))(x)dx = \frac{1}{2} \cdot ((\text{the function}$
 $\text{exp})(\sup A)^2 - (\text{the function exp})(\inf A)^2).$
- (101) $\int_A ((\text{the function exp}) ((\text{the function sin}) + (\text{the function cos}))) (x)dx =$
 $((\text{the function exp}) (\text{the function sin}))(\sup A) - ((\text{the function exp}) (\text{the}$
 $\text{function sin}))(\inf A).$
- (102) $\int_A ((\text{the function exp}) ((\text{the function cos}) - (\text{the function sin}))) (x)dx =$
 $((\text{the function exp}) (\text{the function cos}))(\sup A) - ((\text{the function exp}) (\text{the}$
 $\text{function cos}))(\inf A).$

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