

Basic Properties of the Rank of Matrices over a Field

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Summary. In this paper I present selected properties of triangular matrices and basic properties of the rank of matrices over a field.

I define a submatrix as a matrix formed by selecting certain rows and columns from a bigger matrix. That is in my considerations, as an array, it is cut down to those entries constrained by row and column. Then I introduce the concept of the rank of a $m \times n$ matrix A by the condition: A has the rank r if and only if, there is a $r \times r$ submatrix of A with a non-zero determinant, and for every $k \times k$ submatrix of A with a non-zero determinant we have $k \leq r$.

At the end, I prove that the rank defined by the size of the biggest submatrix with a non-zero determinant of a matrix A , is the same as the maximal number of linearly independent rows of A .

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The articles [27], [10], [37], [23], [1], [2], [12], [38], [39], [7], [8], [3], [4], [24], [36], [31], [15], [6], [13], [28], [14], [41], [30], [19], [34], [42], [9], [22], [16], [11], [25], [40], [18], [20], [26], [33], [21], [17], [35], [32], [29], [43], and [5] provide the terminology and notation for this paper.

1. TRIANGULAR MATRICES

For simplicity, we use the following convention: x, X, Y are sets, D is a non empty set, i, j, k, m, n, m', n' are elements of \mathbb{N} , i_0, j_0, n_0, m_0 are non zero elements of \mathbb{N} , K is a field, a, b are elements of K , p is a finite sequence of elements of K , and M is a matrix over K of dimension n .

Next we state a number of propositions:

- (1) For every matrix A over D of dimension $n \times m$ holds if $n = 0$, then $m = 0$ iff $\text{len } A = n$ and $\text{width } A = m$.
- (2) The following statements are equivalent
 - (i) M is a lower triangular matrix over K of dimension n ,
 - (ii) M^T is an upper triangular matrix over K of dimension n .
- (3) The diagonal of $M =$ the diagonal of M^T .
- (4) Let p_1 be an element of the permutations of n -element set. Suppose $p_1 \neq \text{idseq}(n)$. Then there exists i such that $i \in \text{Seg } n$ and $p_1(i) > i$ and there exists j such that $j \in \text{Seg } n$ and $p_1(j) < j$.
- (5) Let M be a matrix over K of dimension n and p_1 be an element of the permutations of n -element set. Suppose that
 - (i) $p_1 \neq \text{idseq}(n)$, and
 - (ii) M is a lower triangular matrix over K of dimension n or an upper triangular matrix over K of dimension n .
 Then (the product on paths of M)(p_1) = 0_K .
- (6) Let M be a matrix over K of dimension n and I be an element of the permutations of n -element set. If $I = \text{idseq}(n)$, then the diagonal of $M = I\text{-Path } M$.
- (7) Let M be an upper triangular matrix over K of dimension n . Then $\text{Det } M =$ (the multiplication of K) \otimes (the diagonal of M).
- (8) Let M be a lower triangular matrix over K of dimension n . Then $\text{Det } M =$ (the multiplication of K) \otimes (the diagonal of M).
- (9) For every finite set X and for every n holds

$$\overline{\overline{\{Y; Y \text{ ranges over subsets of } X: \text{card } Y = n\}}} = \binom{\text{card } X}{n}.$$
- (10) $\overline{2\text{Set Seg } n} = \binom{n}{2}$.
- (11) Let R be an element of the permutations of n -element set. If $R = \text{Rev}(\text{idseq}(n))$, then R is even iff $\binom{n}{2} \bmod 2 = 0$.
- (12) Let M be a matrix over K of dimension n and R be a permutation of $\text{Seg } n$. Suppose $R = \text{Rev}(\text{idseq}(n))$ and for all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is an upper triangular matrix over K of dimension n .
- (13) Let M be a matrix over K of dimension n and R be a permutation of $\text{Seg } n$. Suppose $R = \text{Rev}(\text{idseq}(n))$ and for all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i + j > n + 1$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is a lower triangular matrix over K of dimension n .
- (14) Let M be a matrix over K of dimension n and R be an element of the permutations of n -element set. Suppose that
 - (i) $R = \text{Rev}(\text{idseq}(n))$, and
 - (ii) for all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$ or for all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i + j > n + 1$

holds $M_{i,j} = 0_K$.

Then $\text{Det } M = (-1)^{\text{sgn}(R)}$ (the multiplication of $K \odot (R\text{-Path } M)$).

- (15) Let M be a matrix over K of dimension n and M_1, M_2 be upper triangular matrices over K of dimension n . Suppose $M = M_1 \cdot M_2$. Then
 - (i) M is an upper triangular matrix over K of dimension n , and
 - (ii) the diagonal of $M = (\text{the diagonal of } M_1) \bullet (\text{the diagonal of } M_2)$.
- (16) Let M be a matrix over K of dimension n and M_1, M_2 be lower triangular matrices over K of dimension n . Suppose $M = M_1 \cdot M_2$. Then
 - (i) M is a lower triangular matrix over K of dimension n , and
 - (ii) the diagonal of $M = (\text{the diagonal of } M_1) \bullet (\text{the diagonal of } M_2)$.

2. THE RANK OF MATRICES

Let D be a non empty set, let M be a matrix over D , let n, m be natural numbers, let n_1 be an element of \mathbb{N}^n , and let m_1 be an element of \mathbb{N}^m . The functor $\text{Segm}(M, n_1, m_1)$ yielding a matrix over D of dimension $n \times m$ is defined as follows:

(Def. 1) For all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of $\text{Segm}(M, n_1, m_1)$ holds $(\text{Segm}(M, n_1, m_1))_{i,j} = M_{n_1(i), m_1(j)}$.

For simplicity, we follow the rules: A denotes a matrix over D , A' denotes a matrix over D of dimension $n' \times m'$, M' denotes a matrix over K of dimension $n' \times m'$, n_1, n_2, n_3 denote elements of \mathbb{N}^n , m_1, m_2 denote elements of \mathbb{N}^m , and M denotes a matrix over K .

Next we state a number of propositions:

- (17) If $\{ \text{rng } n_1, \text{rng } m_1 \} \subseteq$ the indices of A , then $\langle i, j \rangle \in$ the indices of $\text{Segm}(A, n_1, m_1)$ iff $\langle n_1(i), m_1(j) \rangle \in$ the indices of A .
- (18) If $\{ \text{rng } n_1, \text{rng } m_1 \} \subseteq$ the indices of A and $n = 0$ iff $m = 0$, then $(\text{Segm}(A, n_1, m_1))^T = \text{Segm}(A^T, m_1, n_1)$.
- (19) If $\{ \text{rng } n_1, \text{rng } m_1 \} \subseteq$ the indices of A and if $m = 0$, then $n = 0$, then $\text{Segm}(A, n_1, m_1) = (\text{Segm}(A^T, m_1, n_1))^T$.
- (20) For every matrix A over D of dimension 1 holds $A = \langle \langle A_{1,1} \rangle \rangle$.
- (21) If $n = 1$ and $m = 1$, then $\text{Segm}(A, n_1, m_1) = \langle \langle A_{n_1(1), m_1(1)} \rangle \rangle$.
- (22) For every matrix A over D of dimension 2 holds $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$.
- (23) If $n = 2$ and $m = 2$, then $\text{Segm}(A, n_1, m_1) = \begin{pmatrix} A_{n_1(1), m_1(1)} & A_{n_1(1), m_1(2)} \\ A_{n_1(2), m_1(1)} & A_{n_1(2), m_1(2)} \end{pmatrix}$.
- (24) If $i \in \text{Seg } n$ and $\text{rng } m_1 \subseteq \text{Seg width } A$, then $\text{Line}(\text{Segm}(A, n_1, m_1), i) = \text{Line}(A, n_1(i)) \cdot m_1$.

- (25) If $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $n_1(i) = n_1(j)$, then $\text{Line}(\text{Segm}(A, n_1, m_1), i) = \text{Line}(\text{Segm}(A, n_1, m_1), j)$.
- (26) If $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $n_1(i) = n_1(j)$ and $i \neq j$, then $\text{Det Segm}(M, n_1, n_2) = 0_K$.
- (27) If n_1 is not one-to-one, then $\text{Det Segm}(M, n_1, n_2) = 0_K$.
- (28) If $j \in \text{Seg } m$ and $\text{rng } n_1 \subseteq \text{Seg len } A$, then $(\text{Segm}(A, n_1, m_1))_{\square, j} = A_{\square, m_1(j)} \cdot n_1$.
- (29) If $i \in \text{Seg } m$ and $j \in \text{Seg } m$ and $m_1(i) = m_1(j)$, then $(\text{Segm}(A, n_1, m_1))_{\square, i} = (\text{Segm}(A, n_1, m_1))_{\square, j}$.
- (30) If $i \in \text{Seg } m$ and $j \in \text{Seg } m$ and $m_1(i) = m_1(j)$ and $i \neq j$, then $\text{Det Segm}(M, m_2, m_1) = 0_K$.
- (31) If m_1 is not one-to-one, then $\text{Det Segm}(M, m_2, m_1) = 0_K$.
- (32) Let n_1, n_2 be elements of \mathbb{N}^n . Suppose n_1 is one-to-one and n_2 is one-to-one and $\text{rng } n_1 = \text{rng } n_2$. Then there exists a permutation p_1 of $\text{Seg } n$ such that $n_2 = n_1 \cdot p_1$.
- (33) For every function f from $\text{Seg } n$ into $\text{Seg } n$ such that $n_2 = n_1 \cdot f$ holds $\text{Segm}(A, n_2, m_1) = \text{Segm}(A, n_1, m_1) \cdot f$.
- (34) For every function f from $\text{Seg } m$ into $\text{Seg } m$ such that $m_2 = m_1 \cdot f$ holds $(\text{Segm}(A, n_1, m_2))^T = (\text{Segm}(A, n_1, m_1))^T \cdot f$.
- (35) Let p_1 be an element of the permutations of n -element set. If $n_2 = n_3 \cdot p_1$, then $\text{Det Segm}(M, n_2, n_1) = (-1)^{\text{sgn}(p_1)} \text{Det Segm}(M, n_3, n_1)$ and $\text{Det Segm}(M, n_1, n_2) = (-1)^{\text{sgn}(p_1)} \text{Det Segm}(M, n_1, n_3)$.
- (36) For all elements n_1, n_2, n'_1, n'_2 of \mathbb{N}^n such that $\text{rng } n_1 = \text{rng } n'_1$ and $\text{rng } n_2 = \text{rng } n'_2$ holds $\text{Det Segm}(M, n_1, n_2) = \text{Det Segm}(M, n'_1, n'_2)$ or $\text{Det Segm}(M, n_1, n_2) = -\text{Det Segm}(M, n'_1, n'_2)$.
- (37) Let F, F_1 be finite sequences of elements of D and given n_1, m_1 . Suppose $\text{len } F = \text{width } A'$ and $F_1 = F \cdot m_1$ and $\{\text{rng } n_1, \text{rng } m_1\} \subseteq$ the indices of A' . Let given i, j . If $n_1^{-1}(\{j\}) = \{i\}$, then $\text{RLine}(\text{Segm}(A', n_1, m_1), i, F_1) = \text{Segm}(\text{RLine}(A', j, F), n_1, m_1)$.
- (38) Let F be a finite sequence of elements of D and given i, n_1 . If $i \notin \text{rng } n_1$ and $\{\text{rng } n_1, \text{rng } m_1\} \subseteq$ the indices of A' , then $\text{Segm}(A', n_1, m_1) = \text{Segm}(\text{RLine}(A', i, F), n_1, m_1)$.
- (39) If $i \in \text{Seg } n'$ and $i \in \text{rng } n_1$ and $\{\text{rng } n_1, \text{rng } m_1\} \subseteq$ the indices of A' , then there exists n_2 such that $\text{rng } n_2 = (\text{rng } n_1 \setminus \{i\}) \cup \{j\}$ and $\text{Segm}(\text{RLine}(A', i, \text{Line}(A', j)), n_1, m_1) = \text{Segm}(A', n_2, m_1)$.
- (40) For every finite sequence F of elements of D such that $i \notin \text{Seg len } A'$ holds $\text{RLine}(A', i, F) = A'$.

Let n, m be natural numbers, let K be a field, let M be a matrix over K of dimension $n \times m$, and let a be an element of K . Then $a \cdot M$ is a matrix over

K of dimension $n \times m$.

We now state two propositions:

- (41) If $[\text{rng } n_1, \text{rng } m_1] \subseteq$ the indices of M , then $a \cdot \text{Segm}(M, n_1, m_1) = \text{Segm}(a \cdot M, n_1, m_1)$.
- (42) If $n_1 = \text{idseq}(\text{len } A)$ and $m_1 = \text{idseq}(\text{width } A)$, then $\text{Segm}(A, n_1, m_1) = A$.

Let us observe that there exists a subset of \mathbb{N} which is empty, without zero, and finite and there exists a subset of \mathbb{N} which is non empty, without zero, and finite.

Let us consider n . Observe that $\text{Seg } n$ is without zero.

Let X be a without zero set and let Y be a set. One can verify that $X \setminus Y$ is without zero and $X \cap Y$ is without zero.

One can prove the following proposition

- (43) For every finite without zero subset N of \mathbb{N} there exists k such that $N \subseteq \text{Seg } k$.

Let N be a finite without zero subset of \mathbb{N} . Then $\text{Sgm } N$ is an element of $\mathbb{N}^{\text{card } N}$.

Let D be a non empty set, let A be a matrix over D , and let P, Q be without zero finite subsets of \mathbb{N} . The functor $\text{Segm}(A, P, Q)$ yields a matrix over D of dimension $\text{card } P \times \text{card } Q$ and is defined by:

(Def. 2) $\text{Segm}(A, P, Q) = \text{Segm}(A, \text{Sgm } P, \text{Sgm } Q)$.

Next we state two propositions:

- (44) $\text{Segm}(A, \{i_0\}, \{j_0\}) = \langle\langle A_{i_0, j_0} \rangle\rangle$.
- (45) If $i_0 < j_0$ and $n_0 < m_0$, then $\text{Segm}(A, \{i_0, j_0\}, \{n_0, m_0\}) = \begin{pmatrix} A_{i_0, n_0} & A_{i_0, m_0} \\ A_{j_0, n_0} & A_{j_0, m_0} \end{pmatrix}$.

In the sequel P, P_1, P_2, Q, Q_1, Q_2 are without zero finite subsets of \mathbb{N} .

The following propositions are true:

- (46) $\text{Segm}(A, \text{Seg len } A, \text{Seg width } A) = A$.
- (47) If $i \in \text{Seg card } P$ and $Q \subseteq \text{Seg width } A$, then $\text{Line}(\text{Segm}(A, P, Q), i) = \text{Line}(A, (\text{Sgm } P)(i)) \cdot \text{Sgm } Q$.
- (48) If $i \in \text{Seg card } P$, then $\text{Line}(\text{Segm}(A, P, \text{Seg width } A), i) = \text{Line}(A, (\text{Sgm } P)(i))$.
- (49) If $j \in \text{Seg card } Q$ and $P \subseteq \text{Seg len } A$, then $(\text{Segm}(A, P, Q))_{\square, j} = A_{\square, (\text{Sgm } Q)(j)} \cdot \text{Sgm } P$.
- (50) If $j \in \text{Seg card } Q$, then $(\text{Segm}(A, \text{Seg len } A, Q))_{\square, j} = A_{\square, (\text{Sgm } Q)(j)}$.
- (51) $\text{Segm}(A, \text{Seg len } A \setminus \{i\}, \text{Seg width } A) = A_{\setminus i}$.
- (52) $\text{Segm}(M, \text{Seg len } M, \text{Seg width } M \setminus \{i\}) =$ the deleting of i -column in M .
- (53) $(\text{Sgm } P)^{-1}(X)$ is a without zero finite subset of \mathbb{N} .

- (54) If $X \subseteq P$, then $\text{Sgm } X = \text{Sgm } P \cdot \text{Sgm}((\text{Sgm } P)^{-1}(X))$.
- (55) $\{(\text{Sgm } P)^{-1}(X), (\text{Sgm } Q)^{-1}(Y)\} \subseteq$ the indices of $\text{Segm}(A, P, Q)$.
- (56) If $P \subseteq P_1$ and $Q \subseteq Q_1$ and $P_2 = (\text{Sgm } P_1)^{-1}(P)$ and $Q_2 = (\text{Sgm } Q_1)^{-1}(Q)$, then $\{\text{rng Sgm } P_2, \text{rng Sgm } Q_2\} \subseteq$ the indices of $\text{Segm}(A, P_1, Q_1)$ and $\text{Segm}(\text{Segm}(A, P_1, Q_1), P_2, Q_2) = \text{Segm}(A, P, Q)$.
- (57) Suppose $P = \emptyset$ iff $Q = \emptyset$ and $\{P, Q\} \subseteq$ the indices of $\text{Segm}(A, P_1, Q_1)$. Then there exist P_2, Q_2 such that $P_2 \subseteq P_1$ and $Q_2 \subseteq Q_1$ and $P_2 = (\text{Sgm } P_1)^\circ P$ and $Q_2 = (\text{Sgm } Q_1)^\circ Q$ and $\text{card } P_2 = \text{card } P$ and $\text{card } Q_2 = \text{card } Q$ and $\text{Segm}(\text{Segm}(A, P_1, Q_1), P, Q) = \text{Segm}(A, P_2, Q_2)$.
- (58) For every matrix M over K of dimension n holds $\text{Segm}(M, \text{Seg } n \setminus \{i\}, \text{Seg } n \setminus \{j\}) =$ the deleting of i -row and j -column in M .
- (59) Let F, F_2 be finite sequences of elements of D . Suppose $\text{len } F =$ width A' and $F_2 = F \cdot \text{Sgm } Q$ and $\{P, Q\} \subseteq$ the indices of A' . Then $\text{RLine}(\text{Segm}(A', P, Q), i, F_2) = \text{Segm}(\text{RLine}(A', (\text{Sgm } P)(i), F), P, Q)$.
- (60) Let F be a finite sequence of elements of D and given i, P . If $i \notin P$ and $\{P, Q\} \subseteq$ the indices of A' , then $\text{Segm}(A', P, Q) = \text{Segm}(\text{RLine}(A', i, F), P, Q)$.
- (61) If $\{P, Q\} \subseteq$ the indices of A and $\text{card } P = 0$ iff $\text{card } Q = 0$, then $(\text{Segm}(A, P, Q))^T = \text{Segm}(A^T, Q, P)$.
- (62) If $\{P, Q\} \subseteq$ the indices of A and if $\text{card } Q = 0$, then $\text{card } P = 0$, then $\text{Segm}(A, P, Q) = (\text{Segm}(A^T, Q, P))^T$.
- (63) If $\{P, Q\} \subseteq$ the indices of M , then $a \cdot \text{Segm}(M, P, Q) = \text{Segm}(a \cdot M, P, Q)$.

Let D be a non empty set, let A be a matrix over D , and let P, Q be without zero finite subsets of \mathbb{N} . Let us assume that $\text{card } P = \text{card } Q$. The functor $\text{EqSegm}(A, P, Q)$ yields a matrix over D of dimension $\text{card } P$ and is defined by:

(Def. 3) $\text{EqSegm}(A, P, Q) = \text{Segm}(A, P, Q)$.

Next we state several propositions:

- (64) For all P, Q, i, j such that $i \in \text{Seg card } P$ and $j \in \text{Seg card } P$ and $\text{card } P = \text{card } Q$ holds $\text{Delete}(\text{EqSegm}(M, P, Q), i, j) = \text{EqSegm}(M, P \setminus \{(\text{Sgm } P)(i)\}, Q \setminus \{(\text{Sgm } Q)(j)\})$ and $\text{card}(P \setminus \{(\text{Sgm } P)(i)\}) = \text{card}(Q \setminus \{(\text{Sgm } Q)(j)\})$.
- (65) For all M, P, P_1, Q_1 such that $\text{card } P_1 = \text{card } Q_1$ and $P \subseteq P_1$ and $\text{Det EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists Q such that $Q \subseteq Q_1$ and $\text{card } P = \text{card } Q$ and $\text{Det EqSegm}(M, P, Q) \neq 0_K$.
- (66) For all M, P_1, Q, Q_1 such that $\text{card } P_1 = \text{card } Q_1$ and $Q \subseteq Q_1$ and $\text{Det EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists P such that $P \subseteq P_1$ and $\text{card } P = \text{card } Q$ and $\text{Det EqSegm}(M, P, Q) \neq 0_K$.
- (67) If $\text{card } P = \text{card } Q$, then $\{P, Q\} \subseteq$ the indices of A iff $P \subseteq \text{Seg len } A$

and $Q \subseteq \text{Seg width } A$.

- (68) Let given P, Q, i, j_0 . Suppose $i \in \text{Seg } n'$ and $j_0 \in \text{Seg } n'$ and $i \in P$ and $j_0 \notin P$ and $\text{card } P = \text{card } Q$ and $\{P, Q\} \subseteq$ the indices of M' . Then $\text{card } P = \text{card}((P \setminus \{i\}) \cup \{j_0\})$ but $\{(P \setminus \{i\}) \cup \{j_0\}, Q\} \subseteq$ the indices of M' but $\text{Det EqSegm}(\text{RLine}(M', i, \text{Line}(M', j_0)), P, Q) = \text{Det EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$ or $\text{Det EqSegm}(\text{RLine}(M', i, \text{Line}(M', j_0)), P, Q) = -\text{Det EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$.
- (69) If $\text{card } P = \text{card } Q$, then $\{P, Q\} \subseteq$ the indices of A iff $\{Q, P\} \subseteq$ the indices of A^T .
- (70) If $\{P, Q\} \subseteq$ the indices of M and $\text{card } P = \text{card } Q$, then $\text{Det EqSegm}(M, P, Q) = \text{Det EqSegm}(M^T, Q, P)$.
- (71) For every matrix M over K of dimension n holds $\text{Det}(a \cdot M) = \text{power}_K(a, n) \cdot \text{Det } M$.
- (72) If $\{P, Q\} \subseteq$ the indices of M and $\text{card } P = \text{card } Q$, then $\text{Det EqSegm}(a \cdot M, P, Q) = \text{power}_K(a, \text{card } P) \cdot \text{Det EqSegm}(M, P, Q)$.

Let K be a field and let M be a matrix over K . The functor $\text{rk}(M)$ yielding an element of \mathbb{N} is defined by the conditions (Def. 4).

- (Def. 4)(i) There exist P, Q such that $\{P, Q\} \subseteq$ the indices of M and $\text{card } P = \text{card } Q$ and $\text{card } P = \text{rk}(M)$ and $\text{Det EqSegm}(M, P, Q) \neq 0_K$, and
- (ii) for all P_1, Q_1 such that $\{P_1, Q_1\} \subseteq$ the indices of M and $\text{card } P_1 = \text{card } Q_1$ and $\text{Det EqSegm}(M, P_1, Q_1) \neq 0_K$ holds $\text{card } P_1 \leq \text{rk}(M)$.

The following propositions are true:

- (73) For all P, Q such that $\{P, Q\} \subseteq$ the indices of M and $\text{card } P = \text{card } Q$ holds $\text{card } P \leq \text{len } M$ and $\text{card } Q \leq \text{width } M$.
- (74) $\text{rk}(M) \leq \text{len } M$ and $\text{rk}(M) \leq \text{width } M$.
- (75) If $\{\text{rng } n_2, \text{rng } n_3\} \subseteq$ the indices of M and $\text{Det Segm}(M, n_2, n_3) \neq 0_K$, then there exist P_1, P_2 such that $P_1 = \text{rng } n_2$ and $P_2 = \text{rng } n_3$ and $\text{card } P_1 = \text{card } P_2$ and $\text{card } P_1 = n$ and $\text{Det EqSegm}(M, P_1, P_2) \neq 0_K$.
- (76) Let R_1 be an element of \mathbb{N} . Then $\text{rk}(M) = R_1$ if and only if the following conditions are satisfied:
 - (i) there exist elements r_1, r_2 of \mathbb{N}^{R_1} such that $\{\text{rng } r_1, \text{rng } r_2\} \subseteq$ the indices of M and $\text{Det Segm}(M, r_1, r_2) \neq 0_K$, and
 - (ii) for all n, n_2, n_3 such that $\{\text{rng } n_2, \text{rng } n_3\} \subseteq$ the indices of M and $\text{Det Segm}(M, n_2, n_3) \neq 0_K$ holds $n \leq R_1$.
- (77) If $n = 0$ or $m = 0$, then $\text{rk}(\text{Segm}(M, n_1, m_1)) = 0$.
- (78) If $\{\text{rng } n_1, \text{rng } m_1\} \subseteq$ the indices of M , then $\text{rk}(M) \geq \text{rk}(\text{Segm}(M, n_1, m_1))$.
- (79) If $\{P, Q\} \subseteq$ the indices of M , then $\text{rk}(M) \geq \text{rk}(\text{Segm}(M, P, Q))$.
- (80) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $\text{rk}(\text{Segm}(M, P, Q)) \leq \text{rk}(\text{Segm}(M, P_1, Q_1))$.

- (81) For all functions f, g such that $\text{rng } f \subseteq \text{rng } g$ there exists a function h such that $\text{dom } h = \text{dom } f$ and $\text{rng } h \subseteq \text{dom } g$ and $f = g \cdot h$.
- (82) If $[\text{rng } n_1, \text{rng } m_1]$ = the indices of M , then $\text{rk}(M) = \text{rk}(\text{Segm}(M, n_1, m_1))$.
- (83) For every matrix M over K of dimension n holds $\text{rk}(M) = n$ iff $\text{Det } M \neq 0_K$.
- (84) $\text{rk}(M) = \text{rk}(M^T)$.
- (85) For every matrix M over K of dimension $n \times m$ and for every permutation F of $\text{Seg } n$ holds $\text{rk}(M) = \text{rk}(M \cdot F)$.
- (86) If $a \neq 0_K$, then $\text{rk}(M) = \text{rk}(a \cdot M)$.
- (87) Let p, p_2 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\text{rng } f \subseteq \text{dom } p$, then $a \cdot p \cdot f = a \cdot p_2$.
- (88) Let p, p_2, q, q_1 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\text{rng } f \subseteq \text{dom } p$ and $q_1 = q \cdot f$ and $\text{rng } f \subseteq \text{dom } q$, then $(p + q) \cdot f = p_2 + q_1$.
- (89) If $a \neq 0_K$, then $\text{rk}(M') = \text{rk}(\text{RLine}(M', i, a \cdot \text{Line}(M', i)))$.
- (90) If $\text{Line}(M, i) = \text{width } M \mapsto 0_K$, then $\text{rk}(\text{the deleting of } i\text{-row in } M) = \text{rk}(M)$.
- (91) For every p such that $\text{len } p = \text{width } M'$ holds $\text{rk}(\text{the deleting of } i\text{-row in } M') = \text{rk}(\text{RLine}(M', i, 0_K \cdot p))$.
- (92) If $j \in \text{Seg len } M'$ and if $i = j$, then $a \neq -1_K$, then $\text{rk}(M') = \text{rk}(\text{RLine}(M', i, \text{Line}(M', i) + a \cdot \text{Line}(M', j)))$.
- (93) If $j \in \text{Seg len } M'$ and $j \neq i$, then $\text{rk}(\text{the deleting of } i\text{-row in } M') = \text{rk}(\text{RLine}(M', i, a \cdot \text{Line}(M', j)))$.
- (94) $\text{rk}(M) > 0$ iff there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$.

$$(95) \quad \text{rk}(M) = 0 \text{ iff } M = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{(\text{len } M) \times (\text{width } M)}.$$

- (96) $\text{rk}(M) = 1$ if and only if the following conditions are satisfied:
- (i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and
 - (ii) for all i_0, j_0, n_0, m_0 such that $i_0 \neq j_0$ and $n_0 \neq m_0$ and $[\{i_0, j_0\}, \{n_0, m_0\}] \subseteq$ the indices of M holds $\text{Det EqSegm}(M, \{i_0, j_0\}, \{n_0, m_0\}) = 0_K$.
- (97) $\text{rk}(M) = 1$ if and only if the following conditions are satisfied:
- (i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and
 - (ii) for all i, j, n, m such that $[\{i, j\}, \{n, m\}] \subseteq$ the indices of M holds $M_{i,n} \cdot M_{j,m} = M_{i,m} \cdot M_{j,n}$.

- (98) $\text{rk}(M) = 1$ if and only if there exists i such that $i \in \text{Seg len } M$ and there exists j such that $j \in \text{Seg width } M$ and $M_{i,j} \neq 0_K$ and for every k such that $k \in \text{Seg len } M$ there exists a such that $\text{Line}(M, k) = a \cdot \text{Line}(M, i)$.

Let us consider K . Observe that there exists a matrix over K which is diagonal.

One can prove the following propositions:

- (99) Let M be a diagonal matrix over K and N_1 be a set. Suppose $N_1 = \{i : \langle i, i \rangle \in \text{the indices of } M \wedge M_{i,i} \neq 0_K\}$. Let given P, Q . If $\{P, Q\} \subseteq \text{the indices of } M$ and $\text{card } P = \text{card } Q$ and $\text{Det EqSegm}(M, P, Q) \neq 0_K$, then $P \subseteq N_1$ and $Q \subseteq N_1$.
- (100) For every diagonal matrix M over K and for every P such that $\{P, P\} \subseteq \text{the indices of } M$ holds $\text{Segm}(M, P, P)$ is diagonal.
- (101) Let M be a diagonal matrix over K and N_1 be a set. If $N_1 = \{i : \langle i, i \rangle \in \text{the indices of } M \wedge M_{i,i} \neq 0_K\}$, then $\text{rk}(M) = \overline{N_1}$.

For simplicity, we adopt the following rules: v, v_1, v_2, u denote vectors of the n -dimension vector space over K , t, t_1, t_2 denote elements of (the carrier of K) ^{n} , L denotes a linear combination of the n -dimension vector space over K , and M, M_1 denote matrices over K of dimension $m \times n$.

We now state the proposition

- (102)(i) The carrier of the n -dimension vector space over $K = (\text{the carrier of } K)^n$,
- (ii) $0_{\text{the } n\text{-dimension vector space over } K} = n \mapsto 0_K$,
- (iii) if $t_1 = v_1$ and $t_2 = v_2$, then $t_1 + t_2 = v_1 + v_2$, and
- (iv) if $t = v$, then $a \cdot t = a \cdot v$.

Let us consider K, n . Then the n -dimension vector space over K is a strict vector space over K .

Let us consider K, n . One can verify that every vector of the n -dimension vector space over K is function-like and relation-like.

Let us consider K, m, n and let M be a matrix over K of dimension $m \times n$. We introduce $\text{lines}(M)$ as a synonym of $\text{rng } M$. We introduce M is without repeated line as a synonym of M is one-to-one.

Let K be a field, let us consider m, n , and let M be a matrix over K of dimension $m \times n$. Then $\text{lines}(M)$ is a subset of the n -dimension vector space over K .

Next we state two propositions:

- (103) $x \in \text{lines}(M)$ iff there exists i such that $i \in \text{Seg } m$ and $x = \text{Line}(M, i)$.
- (104) Let V be a finite subset of the n -dimension vector space over K . Then there exists a matrix M over K of dimension $\text{card } V \times n$ such that M is without repeated line and $\text{lines}(M) = V$.

Let us consider K , n and let F be a finite sequence of elements of the n -dimension vector space over K . The functor $\text{FinS2MX } F$ yielding a matrix over K of dimension $\text{len } F \times n$ is defined by:

(Def. 5) $\text{FinS2MX } F = F$.

Let us consider K , m , n and let M be a matrix over K of dimension $m \times n$. The functor $\text{MX2FinS } M$ yielding a finite sequence of elements of the n -dimension vector space over K is defined as follows:

(Def. 6) $\text{MX2FinS } M = M$.

One can prove the following propositions:

(105) If $\text{rk}(M) = m$, then M is without repeated line.

(106) If $i \in \text{Seg len } M$ and $a = L(M(i))$, then
 $\text{Line}(\text{FinS2MX}(L \text{ MX2FinS } M), i) = a \cdot \text{Line}(M, i)$.

(107) If M is without repeated line and the support of $L \subseteq \text{lines}(M)$ and $i \in \text{Seg } n$, then $(\sum L)(i) = \sum((\text{FinS2MX}(L \text{ MX2FinS } M))_{\square, i})$.

(108) Let given M, M_1 . Suppose M is without repeated line and for every i such that $i \in \text{Seg } m$ there exists a such that $\text{Line}(M_1, i) = a \cdot \text{Line}(M, i)$. Then there exists a linear combination L of $\text{lines}(M)$ such that $L \text{ MX2FinS } M = M_1$.

(109) Let given M . Suppose M is without repeated line. Then for every i such that $i \in \text{Seg } m$ holds $\text{Line}(M, i) \neq n \mapsto 0_K$ and for every M_1 such that for every i such that $i \in \text{Seg } m$ there exists a such that $\text{Line}(M_1, i) = a \cdot \text{Line}(M, i)$ and for every j such that $j \in \text{Seg } n$ holds $\sum((M_1)_{\square, j}) =$

$$0_K \text{ holds } M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{m \times n} \text{ if and only if } \text{lines}(M) \text{ is linearly}$$

independent.

(110) If $\text{rk}(M) = m$, then $\text{lines}(M)$ is linearly independent.

(111) Let M be a diagonal n -dimensional matrix over K . Suppose $\text{rk}(M) = n$. Then $\text{lines}(M)$ is a basis of the n -dimension vector space over K .

Let us consider K , n . Then the n -dimension vector space over K is a strict finite dimensional vector space over K .

The following propositions are true:

(112) $\dim(\text{the } n\text{-dimension vector space over } K) = n$.

(113) Let given M, i, a . Suppose that for every j such that $j \in \text{Seg } m$ holds $M_{j, i} = a$. Then M is without repeated line if and only if $\text{Segm}(M, \text{Seg len } M, \text{Seg width } M \setminus \{i\})$ is without repeated line.

(114) Let given M, i . Suppose M is without repeated line and $\text{lines}(M)$ is linearly independent and for every j such that $j \in \text{Seg } m$ holds $M_{j, i} = 0_K$. Then $\text{lines}(\text{Segm}(M, \text{Seg len } M, \text{Seg width } M \setminus \{i\}))$ is linearly independent.

- (115) Let V be a vector space over K and U be a finite subset of V . Suppose U is linearly independent. Let u, v be vectors of V . If $u \in U$ and $v \in U$ and $u \neq v$, then $(U \setminus \{u\}) \cup \{u + a \cdot v\}$ is linearly independent.
- (116) Let V be a vector space over K and u, v be vectors of V . Then $x \in \text{Lin}(\{u, v\})$ if and only if there exist a, b such that $x = a \cdot u + b \cdot v$.
- (117) Let given M . Suppose $\text{lines}(M)$ is linearly independent and M is without repeated line. Let given i, j . Suppose $j \in \text{Seg len } M$ and $i \neq j$. Then $\text{RLine}(M, i, \text{Line}(M, i) + a \cdot \text{Line}(M, j))$ is without repeated line and $\text{lines}(\text{RLine}(M, i, \text{Line}(M, i) + a \cdot \text{Line}(M, j)))$ is linearly independent.
- (118) If $P \subseteq \text{Seg } m$, then $\text{lines}(\text{Segm}(M, P, \text{Seg } n)) \subseteq \text{lines}(M)$.
- (119) If $P \subseteq \text{Seg } m$ and $\text{lines}(M)$ is linearly independent, then $\text{lines}(\text{Segm}(M, P, \text{Seg } n))$ is linearly independent.
- (120) If $P \subseteq \text{Seg } m$ and M is without repeated line, then $\text{Segm}(M, P, \text{Seg } n)$ is without repeated line.
- (121) Let M be a matrix over K of dimension $m \times n$. Then $\text{lines}(M)$ is linearly independent and M is without repeated line if and only if $\text{rk}(M) = m$.
- (122) Let U be a subset of the n -dimension vector space over K . Suppose $U \subseteq \text{lines}(M)$. Then there exists P such that $P \subseteq \text{Seg } m$ and $\text{lines}(\text{Segm}(M, P, \text{Seg } n)) = U$ and $\text{Segm}(M, P, \text{Seg } n)$ is without repeated line.
- (123) Let R_1 be an element of \mathbb{N} . Then $\text{rk}(M) = R_1$ if and only if the following conditions are satisfied:
- (i) there exists a finite subset U of the n -dimension vector space over K such that U is linearly independent and $U \subseteq \text{lines}(M)$ and $\text{card } U = R_1$, and
 - (ii) for every finite subset W of the n -dimension vector space over K such that W is linearly independent and $W \subseteq \text{lines}(M)$ holds $\text{card } W \leq R_1$.

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