

Several Classes of BCK-algebras and their Properties

Tao Sun
Qingdao University of Science
and Technology
China

Dahai Hu
Qingdao University of Science
and Technology
China

Xiquan Liang
Qingdao University of Science
and Technology
China

Summary. In this article the general theory of Commutative BCK-algebras and BCI-algebras and several classes of BCK-algebras are given according to [2].

MML identifier: BCIALG_3, version: 7.8.05 4.89.993

The articles [3] and [1] provide the notation and terminology for this paper.

1. THE BASICS OF GENERAL THEORY OF COMMUTATIVE BCK-ALGEBRAS

Let I_1 be a non empty BCI structure with 0. We say that I_1 is commutative if and only if:

(Def. 1) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

Let us observe that BCI-EXAMPLE is commutative.

Let us note that there exists a BCK-algebra which is commutative.

In the sequel X denotes a BCK-algebra and I_1 denotes a non empty subset of X .

We now state a number of propositions:

- (1) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) \leq y \setminus (y \setminus x)$.

- (2) For every BCK-algebra X and for all elements x, y of X holds $x \setminus (x \setminus y) \leq y$ and $x \setminus (x \setminus y) \leq x$.
- (3) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus (y \setminus (y \setminus x))$.
- (4) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.
- (5) X is a commutative BCK-algebra iff for all elements x, y of X such that $x \leq y$ holds $x = y \setminus (y \setminus x)$.
- (6) Let X be a non empty BCI structure with 0 . Then X is a commutative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x)$.
- (7) If X is a commutative BCK-algebra, then for all elements x, y of X such that $x \setminus y = x$ holds $y \setminus x = y$.
- (8) If X is a commutative BCK-algebra, then for all elements x, y, a of X such that $y \leq a$ holds $a \setminus x \setminus (a \setminus y) = y \setminus x$.
- (9) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus y = x$ iff $y \setminus (y \setminus x) = 0_X$.
- (10) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus (y \setminus (y \setminus x)) = x \setminus y$ and $x \setminus y \setminus (x \setminus y \setminus x) = x \setminus y$.
- (11) Suppose X is a commutative BCK-algebra. Let x, y, a be elements of X . If $x \leq a$, then $(a \setminus y) \setminus (a \setminus y \setminus (a \setminus x)) = a \setminus y \setminus (x \setminus y)$.

Let X be a BCI-algebra and let a be an element of X . We say that a is greatest if and only if:

(Def. 2) For every element x of X holds $x \setminus a = 0_X$.

We say that a is positive if and only if:

(Def. 3) $0_X \setminus a = 0_X$.

2. THE BASICS OF GENERAL THEORY OF COMMUTATIVE BCI-ALGEBRAS

Let I_1 be a BCI-algebra. We say that I_1 is BCI-commutative if and only if:

(Def. 4) For all elements x, y of I_1 such that $x \setminus y = 0_{(I_1)}$ holds $x = y \setminus (y \setminus x)$.

We say that I_1 is BCI-weakly-commutative if and only if:

(Def. 5) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (0_{(I_1)} \setminus (x \setminus y)) = y \setminus (y \setminus x)$.

One can check that BCI-EXAMPLE is BCI-commutative and BCI-weakly-commutative.

Let us note that there exists a BCI-algebra which is BCI-commutative and BCI-weakly-commutative.

The following propositions are true:

- (12) For every BCI-algebra X such that there exists an element of X which is greatest holds X is a BCK-algebra.
- (13) Let X be a BCI-algebra. Suppose X is p -semisimple. Then X is BCI-commutative and BCI-weakly-commutative.
- (14) Every commutative BCK-algebra is a BCI-commutative BCI-algebra and a BCI-weakly-commutative BCI-algebra.
- (15) If X is a BCI-weakly-commutative BCI-algebra, then X is BCI-commutative.
- (16) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.
- (17) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus (y \setminus x)) = 0_X \setminus (x \setminus y)$.
- (18) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for every element a of $\text{AtomSet } X$ and for all elements x, y of $\text{Branch } V a$ holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.
- (19) Let X be a non empty BCI structure with 0 . Then X is a BCI-commutative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.
- (20) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq z$ and $z \setminus y \leq z \setminus x$ holds $x \leq y$.
- (21) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq y$ and $x \leq z$ holds $x \leq y \setminus (y \setminus z)$.

3. BOUNDED BCK-ALGEBRAS

Let I_1 be a BCK-algebra. We say that I_1 is bounded if and only if:

(Def. 6) There exists an element of I_1 which is greatest.

Let us note that BCI-EXAMPLE is bounded.

One can verify that there exists a BCK-algebra which is bounded and commutative.

Let I_1 be a bounded BCK-algebra. We say that I_1 is involutory if and only if:

(Def. 7) For every element a of I_1 such that a is greatest and for every element x of I_1 holds $a \setminus (a \setminus x) = x$.

Next we state three propositions:

- (22) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus y = a \setminus y \setminus (a \setminus x)$.

(23) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus (a \setminus y) = y \setminus (a \setminus x)$.

(24) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X such that $x \leq a \setminus y$ holds $y \leq a \setminus x$.

Let I_1 be a BCK-algebra and let a be an element of I_1 . We say that a is Iseki if and only if:

(Def. 8) For every element x of I_1 holds $x \setminus a = 0_{(I_1)}$ and $a \setminus x = a$.

Let I_1 be a BCK-algebra. We say that I_1 is Iseki-extension if and only if:

(Def. 9) There exists an element of I_1 which is Iseki.

Let us observe that BCI-EXAMPLE is Iseki-extension.

Let X be a BCK-algebra. A non empty subset of X is said to be a commutative-ideal of X if:

(Def. 10) $0_X \in \text{it}$ and for all elements x, y, z of X such that $x \setminus y \setminus z \in \text{it}$ and $z \in \text{it}$ holds $x \setminus (y \setminus (y \setminus x)) \in \text{it}$.

The following three propositions are true:

(25) If I_1 is a commutative-ideal of X , then for all elements x, y of X such that $x \setminus y \in I_1$ holds $x \setminus (y \setminus (y \setminus x)) \in I_1$.

(26) For every BCK-algebra X such that I_1 is a commutative-ideal of X holds I_1 is an ideal of X .

(27) If I_1 is a commutative-ideal of X , then for all elements x, y of X such that $x \setminus (x \setminus y) \in I_1$ holds $y \setminus (y \setminus x) \setminus (x \setminus y) \in I_1$.

4. IMPLICATIVE AND POSITIVE-IMPLICATIVE BCK-ALGEBRAS

Let I_1 be a BCK-algebra. We say that I_1 is BCK-positive-implicative if and only if:

(Def. 11) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus z \setminus (y \setminus z)$.

We say that I_1 is BCK-implicative if and only if:

(Def. 12) For all elements x, y of I_1 holds $x \setminus (y \setminus x) = x$.

Let us observe that BCI-EXAMPLE is BCK-positive-implicative and BCK-implicative.

Let us mention that there exists a BCK-algebra which is Iseki-extension, BCK-positive-implicative, BCK-implicative, bounded, and commutative.

The following propositions are true:

(28) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y$.

- (29) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = x \setminus (x \setminus (y \setminus (y \setminus x)))$.
- (30) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus (x \setminus (x \setminus y))$.
- (31) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus z \setminus (y \setminus z) \leq (x \setminus y) \setminus z$.
- (32) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y \leq x \setminus y \setminus y$.
- (33) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $x \setminus (x \setminus (y \setminus (y \setminus x))) \leq (x \setminus (x \setminus y)) \setminus (y \setminus x)$.
- (34) X is a BCK-implicative BCK-algebra if and only if X is a commutative BCK-algebra and a BCK-positive-implicative BCK-algebra.
- (35) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus x)$.
- (36) Let X be a non empty BCI structure with 0 . Then X is a BCK-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x) \setminus (x \setminus y)$.
- (37) Let X be a bounded BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if X is involutory and BCK-positive-implicative.
- (38) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus (y \setminus x)) = 0_X$.
- (39) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.
- (40) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x \setminus z)$.
- (41) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $x \setminus (x \setminus (y \setminus z)) = (y \setminus z) \setminus (y \setminus z \setminus (x \setminus z))$.
- (42) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = (y \setminus (y \setminus x)) \setminus (x \setminus y)$.
- (43) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus (a \setminus x \setminus x) = 0_X$.
- (44) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $x \setminus (a \setminus x) = x$.
- (45) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus x = a \setminus x$.

- (46) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $a \setminus y \setminus (a \setminus y \setminus x) = x \setminus y$.
- (47) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $y \setminus (y \setminus x) = x \setminus (a \setminus y)$.
- (48) Let X be a bounded commutative BCK-algebra and a be an element of X . Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y, z of X holds $(x \setminus (y \setminus z)) \setminus (x \setminus y) \leq x \setminus (a \setminus z)$.

REFERENCES

- [1] Yuzhong Ding. Several classes of BCI-algebras and their properties. *Formalized Mathematics*, 15(1):1–9, 2007.
- [2] Jie Meng and YoungLin Liu. *An Introduction to BCI-algebras*. Shaanxi Scientific and Technological Press, 2001.
- [3] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

Received September 19, 2007
