

Gauss Lemma and Law of Quadratic Reciprocity

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Summary. In this paper, we defined the quadratic residue and proved its fundamental properties on the base of some useful theorems. Then we defined the Legendre symbol and proved its useful theorems [14], [12]. Finally, Gauss Lemma and Law of Quadratic Reciprocity are proven.

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The papers [20], [10], [9], [11], [4], [1], [2], [17], [8], [19], [7], [16], [13], [21], [22], [5], [18], [3], [15], [6], and [23] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: $i, i_1, i_2, i_3, j, a, b, x$ denote integers, d, e, n denote natural numbers, f, f' denote finite sequences of elements of \mathbb{Z} , g, g_1, g_2 denote finite sequences of elements of \mathbb{R} , and p denotes a prime number.

We now state two propositions:

- (1) If $i_1 \mid i_2$ and $i_1 \mid i_3$, then $i_1 \mid i_2 - i_3$.
- (2) If $i \mid a$ and $i \mid a - b$, then $i \mid b$.

Let us consider f . The functor $\mathcal{P}_{\mathbb{Z}}(f)$ yields a function from \mathbb{Z} into \mathbb{Z} and is defined by the condition (Def. 1).

(Def. 1) Let x be an element of \mathbb{Z} . Then there exists a finite sequence f' of elements of \mathbb{Z} such that $\text{len } f' = \text{len } f$ and for every d such that $d \in \text{dom } f'$ holds $f'(d) = f(d) \cdot x^{d-1}$ and $(\mathcal{P}_{\mathbb{Z}}(f))(x) = \sum f'$.

Let f be a finite sequence of elements of \mathbb{Z} and let x be an integer. Observe that $(\mathcal{P}_{\mathbb{Z}}(f))(x)$ is integer.

We now state two propositions:

- (3) If $\text{len } f = 1$, then $\mathcal{P}_{\mathbb{Z}}(f) = \mathbb{Z} \mapsto f(1)$.
- (4) If $\text{len } f = 1$, then for every element x of \mathbb{Z} holds $(\mathcal{P}_{\mathbb{Z}}(f))(x) = f(1)$.

In the sequel f' denotes a finite sequence of elements of \mathbb{R} .

Next we state three propositions:

- (5) Let given g, g_1, g_2 . Suppose $\text{len } g = n + 1$ and $\text{len } g_1 = \text{len } g$ and $\text{len } g_2 = \text{len } g$ and for every d such that $d \in \text{dom } g$ holds $g(d) = g_1(d) - g_2(d)$. Then there exists f' such that $\text{len } f' = \text{len } g - 1$ and for every d such that $d \in \text{dom } f'$ holds $f'(d) = g_1(d) - g_2(d+1)$ and $\sum g = ((\sum f') + g_1(n+1)) - g_2(1)$.
- (6) Suppose $\text{len } f = n + 2$. Let a be an integer. Then there exists a finite sequence f' of elements of \mathbb{Z} and there exists an integer r such that $\text{len } f' = n+1$ and for every element x of \mathbb{Z} holds $(\mathcal{P}_{\mathbb{Z}}(f))(x) = (x-a) \cdot (\mathcal{P}_{\mathbb{Z}}(f'))(x) + r$ and $f(n+2) = f'(n+1)$.
- (7) If $p \mid i \cdot j$, then $p \mid i$ or $p \mid j$.

In the sequel f', g are finite sequences of elements of \mathbb{Z} .

The following proposition is true

- (8) Let given f . Suppose $\text{len } f = n+1$ and $p > 2$ and $p \nmid f(n+1)$. Let given f' . Suppose for every d such that $d \in \text{dom } f'$ holds $(\mathcal{P}_{\mathbb{Z}}(f))(f'(d)) \pmod p = 0$ and for all d, e such that $d, e \in \text{dom } f'$ and $d \neq e$ holds $f'(d) \not\equiv f'(e) \pmod p$. Then $\text{len } f' \leq n$.

Let a be an integer and let m be a natural number. We say that a is quadratic residue mod m if and only if:

(Def. 2) There exists an integer x such that $(x^2 - a) \pmod m = 0$.

In the sequel b, m denote natural numbers.

We now state four propositions:

- (9) If $\text{gcd } m = 1$, then a^2 is quadratic residue mod m .
- (10) 1 is quadratic residue mod 2.
- (11) If $i \pmod m = 1$ and i is quadratic residue mod m and $i \equiv j \pmod m$, then j is quadratic residue mod m .
- (12) If $i \mid j$, then $i \pmod j = |i|$.

Let k be an integer and let a be a natural number. One can verify that k^a is integer.

One can prove the following propositions:

- (13) For all integers i, j, m such that $i \bmod m = j \bmod m$ holds $i^n \bmod m = j^n \bmod m$.
- (14) If $a \gcd p = 1$ and $(x^2 - a) \bmod p = 0$, then x and p are relative prime.
- (15) Suppose $p > 2$ and $a \gcd p = 1$ and a is quadratic residue mod p . Then there exist integers x, y such that $(x^2 - a) \bmod p = 0$ and $(y^2 - a) \bmod p = 0$ and $x \not\equiv y \pmod{p}$.

Let f be a finite sequence of elements of \mathbb{N} and let us consider d . One can check that $f(d)$ is natural.

The following propositions are true:

- (16) Suppose $p > 2$. Then there exists a finite sequence f of elements of \mathbb{N} such that
 - (i) $\text{len } f = (p - 1) \div 2$,
 - (ii) for every d such that $d \in \text{dom } f$ holds $\gcd(f(d), p) = 1$,
 - (iii) for every d such that $d \in \text{dom } f$ holds $f(d)$ is quadratic residue mod p , and
 - (iv) for all d, e such that $d, e \in \text{dom } f$ and $d \neq e$ holds $f(d) \not\equiv f(e) \pmod{p}$.
- (17) If $p > 2$ and $a \gcd p = 1$ and a is quadratic residue mod p , then $a^{(p-1) \div 2} \bmod p = 1$.
- (18) If $p > 2$ and $b \gcd p = 1$ and b is not quadratic residue mod p , then $b^{(p-1) \div 2} \bmod p = p - 1$.
- (19) If $p > 2$ and $a \gcd p = 1$ and a is not quadratic residue mod p , then $a^{(p-1) \div 2} \bmod p = p - 1$.
- (20) If $p > 2$ and $a \gcd p = 1$ and a is quadratic residue mod p , then $(a^{(p-1) \div 2} - 1) \bmod p = 0$.
- (21) If $p > 2$ and $a \gcd p = 1$ and a is not quadratic residue mod p , then $(a^{(p-1) \div 2} + 1) \bmod p = 0$.

In the sequel b is an integer.

We now state three propositions:

- (22) Suppose $p > 2$ and $a \gcd p = 1$ and $b \gcd p = 1$ and a is quadratic residue mod p and b is quadratic residue mod p . Then $a \cdot b$ is quadratic residue mod p .
- (23) Suppose $p > 2$ and $a \gcd p = 1$ and $b \gcd p = 1$ and a is quadratic residue mod p and b is not quadratic residue mod p . Then $a \cdot b$ is not quadratic residue mod p .
- (24) Suppose $p > 2$ and $a \gcd p = 1$ and $b \gcd p = 1$ and a is not quadratic residue mod p and b is not quadratic residue mod p . Then $a \cdot b$ is quadratic residue mod p .

Let a be an integer and let p be a prime number. The functor $\left(\frac{a}{p}\right)$ yielding an integer is defined by:

$$\text{(Def. 3)} \quad \left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is quadratic residue mod } p, \\ -1, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

$$(25) \quad \left(\frac{a}{p}\right) = 1 \text{ or } \left(\frac{a}{p}\right) = -1.$$

$$(26) \quad \text{If } a \gcd p = 1, \text{ then } \left(\frac{a^2}{p}\right) = 1.$$

$$(27) \quad \left(\frac{1}{p}\right) = 1.$$

$$(28) \quad \text{If } p > 2 \text{ and } a \gcd p = 1, \text{ then } \left(\frac{a}{p}\right) \equiv a^{(p-1) \div 2} \pmod{p}.$$

$$(29) \quad \text{If } p > 2 \text{ and } a \gcd p = 1 \text{ and } a \equiv b \pmod{p}, \text{ then } \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

$$(30) \quad \text{If } p > 2 \text{ and } a \gcd p = 1 \text{ and } b \gcd p = 1, \text{ then } \left(\frac{a \cdot b}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right).$$

$$(31) \quad \text{If for every } d \text{ such that } d \in \text{dom } f' \text{ holds } f'(d) = 1 \text{ or } f'(d) = -1, \text{ then } \prod f' = 1 \text{ or } \prod f' = -1.$$

In the sequel m denotes an integer.

One can prove the following propositions:

$$(32) \quad \text{For all } g, f' \text{ such that } \text{len } g = \text{len } f' \text{ and for every } d \text{ such that } d \in \text{dom } g \text{ holds } g(d) \equiv f'(d) \pmod{m} \text{ holds } \prod g \equiv \prod f' \pmod{m}.$$

$$(33) \quad \text{For all } g, f' \text{ such that } \text{len } g = \text{len } f' \text{ and for every } d \text{ such that } d \in \text{dom } g \text{ holds } g(d) \equiv -f'(d) \pmod{m} \text{ holds } \prod g \equiv (-1)^{\text{len } g} \cdot \prod f' \pmod{m}.$$

In the sequel f denotes a finite sequence of elements of \mathbb{N} .

Next we state several propositions:

$$(34) \quad \text{Suppose } p > 2 \text{ and for every } d \text{ such that } d \in \text{dom } f \text{ holds } \gcd(f(d), p) = 1. \text{ Then there exists a finite sequence } f' \text{ of elements of } \mathbb{Z} \text{ such that } \text{len } f' = \text{len } f \text{ and for every } d \text{ such that } d \in \text{dom } f' \text{ holds } f'(d) = \left(\frac{f(d)}{p}\right) \text{ and } \left(\frac{\prod f}{p}\right) = \prod f'.$$

$$(35) \quad \text{If } p > 2 \text{ and } \gcd(d, p) = 1 \text{ and } \gcd(e, p) = 1, \text{ then } \left(\frac{d^2 \cdot e}{p}\right) = \left(\frac{e}{p}\right).$$

$$(36) \quad \text{If } p > 2, \text{ then } \left(\frac{-1}{p}\right) = (-1)^{(p-1) \div 2}.$$

$$(37) \quad \text{If } p > 2 \text{ and } p \bmod 4 = 1, \text{ then } -1 \text{ is quadratic residue mod } p.$$

$$(38) \quad \text{If } p > 2 \text{ and } p \bmod 4 = 3, \text{ then } -1 \text{ is not quadratic residue mod } p.$$

$$(39) \quad \text{Let } D \text{ be a non empty set, } g \text{ be a finite sequence of elements of } D, \text{ and } i, j \text{ be natural numbers. Then } g \text{ is one-to-one if and only if } \text{Swap}(g, i, j) \text{ is one-to-one.}$$

$$(40) \quad \text{Let } g \text{ be a finite sequence of elements of } \mathbb{N}. \text{ Suppose } \text{len } g = n \text{ and for every } d \text{ such that } d \in \text{dom } g \text{ holds } g(d) > 0 \text{ and } g(d) \leq n \text{ and } g \text{ is one-to-one. Then } \text{rng } g = \text{Seg } n.$$

In the sequel a, m are natural numbers.

Next we state several propositions:

- (41) Let g be a finite sequence of elements of \mathbb{N} . Suppose $p > 2$ and $\gcd(a, p) = 1$ and $g = a \cdot \text{idseq}((p - 1) \div 2)$ and $m = \overline{\{k \in \mathbb{N}: k \in \text{rng}(g \bmod p) \wedge k > \frac{p}{2}\}}$. Then $\left(\frac{a}{p}\right) = (-1)^m$.
- (42) If $p > 2$, then $\left(\frac{2}{p}\right) = (-1)^{(p^2-1) \div 8}$.
- (43) If $p > 2$ and if $p \bmod 8 = 1$ or $p \bmod 8 = 7$, then 2 is quadratic residue mod p .
- (44) If $p > 2$ and if $p \bmod 8 = 3$ or $p \bmod 8 = 5$, then 2 is not quadratic residue mod p .
- (45) For all natural numbers a, b such that $a \bmod 2 = b \bmod 2$ holds $(-1)^a = (-1)^b$.

In the sequel g, h, k denote finite sequences of elements of \mathbb{R} .

Next we state two propositions:

- (46) If $\text{len } g = \text{len } h$ and $\text{len } g = \text{len } k$, then $g \wedge g - h \wedge k = (g - h) \wedge (g - k)$.
- (47) For every finite sequence g of elements of \mathbb{R} and for every real number m holds $\sum(\text{len } g \mapsto m - g) = \text{len } g \cdot m - \sum g$.

In the sequel X denotes a finite set and F denotes a finite sequence of elements of 2^X .

Let us consider X, F . Then \overline{F} is a cardinal yielding finite sequence of elements of \mathbb{N} .

The following proposition is true

- (48) Let g be a finite sequence of elements of 2^X . Suppose $\text{len } g = n$ and for all d, e such that $d, e \in \text{dom } g$ and $d \neq e$ holds $g(d)$ misses $g(e)$. Then $\overline{\bigcup \text{rng } g} = \sum \overline{g}$.

In the sequel q is a prime number.

The following three propositions are true:

- (49) If $p > 2$ and $q > 2$ and $p \neq q$, then $\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{(p-1) \div 2 \cdot ((q-1) \div 2)}$.
- (50) If $p > 2$ and $q > 2$ and $p \neq q$ and $p \bmod 4 = 3$ and $q \bmod 4 = 3$, then $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.
- (51) If $p > 2$ and $q > 2$ and $p \neq q$ and $p \bmod 4 = 1$ or $q \bmod 4 = 1$, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.

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