

## Difference and Difference Quotient. Part II

Bo Li  
Qingdao University of Science  
and Technology  
China

Yanping Zhuang  
and Technology  
China

Xiquan Liang  
Qingdao University of Science  
and Technology  
China

**Summary.** In this article, we give some important properties of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions [11].

MML identifier: DIFF\_2, version: 7.8.09 4.97.1001

The articles [8], [1], [4], [2], [3], [5], [7], [12], [13], [6], [9], and [10] provide the notation and terminology for this paper.

We follow the rules:  $h, r, r_1, r_2, x_0, x_1, x_2, x_3, x_4, x_5, x, a, b, c, k$  denote real numbers and  $f, f_1, f_2$  denote functions from  $\mathbb{R}$  into  $\mathbb{R}$ .

Next we state a number of propositions:

- (1)<sup>1</sup>  $\Delta[f](x, x+h) = \frac{(\bar{\Delta}_h[f])(1)(x)}{h}$ .
- (2) If  $h \neq 0$ , then  $\Delta[f](x, x+h, x+2 \cdot h) = \frac{(\bar{\Delta}_h[f])(2)(x)}{2 \cdot h^2}$ .
- (3)  $\Delta[f](x-h, x) = \frac{(\bar{\nabla}_h[f])(1)(x)}{h}$ .
- (4) If  $h \neq 0$ , then  $\Delta[f](x-2 \cdot h, x-h, x) = \frac{(\bar{\nabla}_h[f])(2)(x)}{2 \cdot h^2}$ .
- (5)  $\Delta[r f](x_0, x_1, x_2) = r \cdot \Delta[f](x_0, x_1, x_2)$ .
- (6)  $\Delta[f_1 + f_2](x_0, x_1, x_2) = \Delta[f_1](x_0, x_1, x_2) + \Delta[f_2](x_0, x_1, x_2)$ .

---

<sup>1</sup>The notation  $\Delta(f, x, y)$  has been changed to  $\Delta[f](x, y)$ . More in Addenda.

- (7)  $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2)$ .  
(8)  $\Delta[r f](x_0, x_1, x_2, x_3) = r \cdot \Delta[f](x_0, x_1, x_2, x_3)$ .  
(9)  $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3) = \Delta[f_1](x_0, x_1, x_2, x_3) + \Delta[f_2](x_0, x_1, x_2, x_3)$ .  
(10)  $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3)$ .

Let  $f$  be a real-yielding function and let  $x_0, x_1, x_2, x_3, x_4$  be real numbers. The functor  $\Delta[f](x_0, x_1, x_2, x_3, x_4)$  yielding a real number is defined as follows:

$$\text{(Def. 1)} \quad \Delta[f](x_0, x_1, x_2, x_3, x_4) = \frac{\Delta[f](x_0, x_1, x_2, x_3) - \Delta[f](x_1, x_2, x_3, x_4)}{x_0 - x_4}.$$

Next we state three propositions:

- (11)  $\Delta[r f](x_0, x_1, x_2, x_3, x_4) = r \cdot \Delta[f](x_0, x_1, x_2, x_3, x_4)$ .  
(12)  $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3, x_4) = \Delta[f_1](x_0, x_1, x_2, x_3, x_4) + \Delta[f_2](x_0, x_1, x_2, x_3, x_4)$ .  
(13)  $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3, x_4) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3, x_4) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3, x_4)$ .

Let  $f$  be a real-yielding function and let  $x_0, x_1, x_2, x_3, x_4, x_5$  be real numbers. The functor  $\Delta[f](x_0, x_1, x_2, x_3, x_4, x_5)$  yields a real number and is defined as follows:

$$\text{(Def. 2)} \quad \Delta[f](x_0, x_1, x_2, x_3, x_4, x_5) = \frac{\Delta[f](x_0, x_1, x_2, x_3, x_4) - \Delta[f](x_1, x_2, x_3, x_4, x_5)}{x_0 - x_5}.$$

We now state a number of propositions:

- (14)  $\Delta[r f](x_0, x_1, x_2, x_3, x_4, x_5) = r \cdot \Delta[f](x_0, x_1, x_2, x_3, x_4, x_5)$ .  
(15)  $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3, x_4, x_5) = \Delta[f_1](x_0, x_1, x_2, x_3, x_4, x_5) + \Delta[f_2](x_0, x_1, x_2, x_3, x_4, x_5)$ .  
(16)  $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3, x_4, x_5) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3, x_4, x_5) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3, x_4, x_5)$ .  
(17) If  $x_0, x_1, x_2$  are mutually different, then  $\Delta[f](x_0, x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1) \cdot (x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0) \cdot (x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0) \cdot (x_2 - x_1)}$ .  
(18) If  $x_0, x_1, x_2, x_3$  are mutually different, then  $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_2, x_3, x_0)$  and  $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_3, x_2, x_1, x_0)$ .  
(19) If  $x_0, x_1, x_2, x_3$  are mutually different, then  $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_0, x_2, x_3)$  and  $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_2, x_0, x_3)$ .  
(20) If  $f$  is constant, then  $\Delta[f](x_0, x_1, x_2) = 0$ .  
(21) If  $x_0 \neq x_1$ , then  $\Delta[a \square + b](x_0, x_1) = a$ .  
(22) If  $x_0, x_1, x_2$  are mutually different, then  $\Delta[a \square + b](x_0, x_1, x_2) = 0$ .  
(23) If  $x_0, x_1, x_2, x_3$  are mutually different, then  $\Delta[a \square + b](x_0, x_1, x_2, x_3) = 0$ .  
(24) For every  $x$  holds  $(\Delta_h[a \square + b])(x) = a \cdot h$ .  
(25) For every  $x$  holds  $(\nabla_h[a \square + b])(x) = a \cdot h$ .  
(26) For every  $x$  holds  $(\delta_h[a \square + b])(x) = a \cdot h$ .

- (27) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$  and  $x_0 \neq x_1$ , then  $\Delta[f](x_0, x_1) = a \cdot (x_0 + x_1) + b$ .
- (28) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$  and  $x_0, x_1, x_2$  are mutually different, then  $\Delta[f](x_0, x_1, x_2) = a$ .
- (29) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$  and  $x_0, x_1, x_2, x_3$  are mutually different, then  $\Delta[f](x_0, x_1, x_2, x_3) = 0$ .
- (30) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$  and  $x_0, x_1, x_2, x_3, x_4$  are mutually different, then  $\Delta[f](x_0, x_1, x_2, x_3, x_4) = 0$ .
- (31) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$ , then for every  $x$  holds  $(\Delta_h[f])(x) = 2 \cdot a \cdot h \cdot x + a \cdot h^2 + b \cdot h$ .
- (32) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$ , then for every  $x$  holds  $(\nabla_h[f])(x) = (2 \cdot a \cdot h \cdot x - a \cdot h^2) + b \cdot h$ .
- (33) If for every  $x$  holds  $f(x) = a \cdot x^2 + b \cdot x + c$ , then for every  $x$  holds  $(\delta_h[f])(x) = 2 \cdot a \cdot h \cdot x + b \cdot h$ .
- (34) If for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x_0 \neq x_1$  and  $x_0 \neq 0$  and  $x_1 \neq 0$ , then  $\Delta[f](x_0, x_1) = -\frac{k}{x_0 \cdot x_1}$ .
- (35) If for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x_0 \neq 0$  and  $x_1 \neq 0$  and  $x_2 \neq 0$  and  $x_0, x_1, x_2$  are mutually different, then  $\Delta[f](x_0, x_1, x_2) = \frac{k}{x_0 \cdot x_1 \cdot x_2}$ .
- (36) Suppose for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x_0 \neq 0$  and  $x_1 \neq 0$  and  $x_2 \neq 0$  and  $x_3 \neq 0$  and  $x_0, x_1, x_2, x_3$  are mutually different. Then  $\Delta[f](x_0, x_1, x_2, x_3) = -\frac{k}{x_0 \cdot x_1 \cdot x_2 \cdot x_3}$ .
- (37) Suppose for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x_0 \neq 0$  and  $x_1 \neq 0$  and  $x_2 \neq 0$  and  $x_3 \neq 0$  and  $x_4 \neq 0$  and  $x_0, x_1, x_2, x_3, x_4$  are mutually different. Then  $\Delta[f](x_0, x_1, x_2, x_3, x_4) = \frac{k}{x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4}$ .
- (38) If for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x \neq 0$  and  $x + h \neq 0$ , then for every  $x$  holds  $(\Delta_h[f])(x) = \frac{-k \cdot h}{(x+h) \cdot x}$ .
- (39) If for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x \neq 0$  and  $x - h \neq 0$ , then for every  $x$  holds  $(\nabla_h[f])(x) = \frac{-k \cdot h}{(x-h) \cdot x}$ .
- (40) If for every  $x$  holds  $f(x) = \frac{k}{x}$  and  $x + \frac{h}{2} \neq 0$  and  $x - \frac{h}{2} \neq 0$ , then for every  $x$  holds  $(\delta_h[f])(x) = \frac{-k \cdot h}{(x - \frac{h}{2}) \cdot (x + \frac{h}{2})}$ .
- (41)  $\Delta[\text{the function } \sin](x_0, x_1) = \frac{2 \cdot \cos(\frac{x_0+x_1}{2}) \cdot \sin(\frac{x_0-x_1}{2})}{x_0-x_1}$ .
- (42) For every  $x$  holds  $(\Delta_h[\text{the function } \sin])(x) = 2 \cdot (\cos(\frac{2 \cdot x+h}{2}) \cdot \sin(\frac{h}{2}))$ .
- (43) For every  $x$  holds  $(\nabla_h[\text{the function } \sin])(x) = 2 \cdot (\cos(\frac{2 \cdot x-h}{2}) \cdot \sin(\frac{h}{2}))$ .
- (44) For every  $x$  holds  $(\delta_h[\text{the function } \sin])(x) = 2 \cdot (\cos x \cdot \sin(\frac{h}{2}))$ .
- (45)  $\Delta[\text{the function } \cos](x_0, x_1) = -\frac{2 \cdot \sin(\frac{x_0+x_1}{2}) \cdot \sin(\frac{x_0-x_1}{2})}{x_0-x_1}$ .
- (46) For every  $x$  holds  $(\Delta_h[\text{the function } \cos])(x) = -2 \cdot (\sin(\frac{2 \cdot x+h}{2}) \cdot \sin(\frac{h}{2}))$ .
- (47) For every  $x$  holds  $(\nabla_h[\text{the function } \cos])(x) = -2 \cdot (\sin(\frac{2 \cdot x-h}{2}) \cdot \sin(\frac{h}{2}))$ .

- (48) For every  $x$  holds  $(\delta_h[\text{the function cos}])(x) = -2 \cdot (\sin x \cdot \sin(\frac{h}{2}))$ .
- (49)  $\Delta[(\text{the function sin}) (\text{the function sin})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x_1) - \cos(2 \cdot x_0))}{x_0 - x_1}$ .
- (50) For every  $x$  holds  $(\Delta_h[(\text{the function sin}) (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (x + h)))$ .
- (51) For every  $x$  holds  $(\nabla_h[(\text{the function sin}) (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot (x - h)) - \cos(2 \cdot x))$ .
- (52) For every  $x$  holds  $(\delta_h[(\text{the function sin}) (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x - h) - \cos(2 \cdot x + h))$ .
- (53)  $\Delta[(\text{the function sin}) (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\sin(2 \cdot x_0) - \sin(2 \cdot x_1))}{x_0 - x_1}$ .
- (54) For every  $x$  holds  $(\Delta_h[(\text{the function sin}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot (x + h)) - \sin(2 \cdot x))$ .
- (55) For every  $x$  holds  $(\nabla_h[(\text{the function sin}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot x) - \sin(2 \cdot (x - h)))$ .
- (56) For every  $x$  holds  $(\delta_h[(\text{the function sin}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot x + h) - \sin(2 \cdot x - h))$ .
- (57)  $\Delta[(\text{the function cos}) (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x_0) - \cos(2 \cdot x_1))}{x_0 - x_1}$ .
- (58) For every  $x$  holds  $(\Delta_h[(\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot (x + h)) - \cos(2 \cdot x))$ .
- (59) For every  $x$  holds  $(\nabla_h[(\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (x - h)))$ .
- (60) For every  $x$  holds  $(\delta_h[(\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x + h) - \cos(2 \cdot x - h))$ .
- (61)  $\Delta[(\text{the function sin}) (\text{the function sin}) (\text{the function cos})](x_0, x_1) = \frac{-\frac{1}{2} \cdot (\sin(\frac{3 \cdot (x_1 + x_0)}{2}) \cdot \sin(\frac{3 \cdot (x_1 - x_0)}{2}) + \sin(\frac{x_0 + x_1}{2}) \cdot \sin(\frac{x_0 - x_1}{2}))}{x_0 - x_1}$ .
- (62) Let given  $x$ . Then  $(\Delta_h[(\text{the function sin}) (\text{the function sin}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(\frac{6 \cdot x + 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}) - \sin(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{h}{2}))$ .
- (63) Let given  $x$ . Then  $(\nabla_h[(\text{the function sin}) (\text{the function sin}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(\frac{6 \cdot x - 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2})) - \frac{1}{2} \cdot (\sin(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}))$ .
- (64) For every  $x$  holds  $(\delta_h[(\text{the function sin}) (\text{the function sin}) (\text{the function cos})])(x) = -\frac{1}{2} \cdot (\sin x \cdot \sin(\frac{h}{2})) + \frac{1}{2} \cdot (\sin(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$ .
- (65)  $\Delta[(\text{the function sin}) (\text{the function cos}) (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(\frac{x_0 + x_1}{2}) \cdot \sin(\frac{x_0 - x_1}{2}) + \cos(\frac{3 \cdot (x_0 + x_1)}{2}) \cdot \sin(\frac{3 \cdot (x_0 - x_1)}{2}))}{x_0 - x_1}$ .
- (66) Let given  $x$ . Then  $(\Delta_h[(\text{the function sin}) (\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{h}{2}) + \cos(\frac{6 \cdot x + 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$ .
- (67) Let given  $x$ . Then  $(\nabla_h[(\text{the function sin}) (\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}) + \cos(\frac{6 \cdot x - 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$ .

- (68) For every  $x$  holds  $(\delta_h[(\text{the function sin}) (\text{the function cos}) (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos x \cdot \sin(\frac{h}{2}) + \cos(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$ .
- (69) If  $x_0 \in \text{dom}(\text{the function tan})$  and  $x_1 \in \text{dom}(\text{the function tan})$ , then  $\Delta[\text{the function tan}](x_0, x_1) = \frac{\sin(x_0 - x_1)}{\cos x_0 \cdot \cos x_1 \cdot (x_0 - x_1)}$ .
- (70) If  $x_0 \in \text{dom}(\text{the function cot})$  and  $x_1 \in \text{dom}(\text{the function cot})$ , then  $\Delta[\text{the function cot}](x_0, x_1) = -\frac{\sin(x_0 - x_1)}{\sin x_0 \cdot \sin x_1 \cdot (x_0 - x_1)}$ .
- (71) Suppose  $x_0 \in \text{dom}(\text{the function cosec})$  and  $x_1 \in \text{dom}(\text{the function cosec})$ . Then  $\Delta[\text{the function cosec}](x_0, x_1) = \frac{2 \cdot \cos(\frac{x_1 + x_0}{2}) \cdot \sin(\frac{x_1 - x_0}{2})}{\sin x_1 \cdot \sin x_0 \cdot (x_0 - x_1)}$ .
- (72) Suppose  $x_0 \in \text{dom}(\text{the function sec})$  and  $x_1 \in \text{dom}(\text{the function sec})$ . Then  $\Delta[\text{the function sec}](x_0, x_1) = -\frac{2 \cdot \sin(\frac{x_1 + x_0}{2}) \cdot \sin(\frac{x_1 - x_0}{2})}{\cos x_1 \cdot \cos x_0 \cdot (x_0 - x_1)}$ .

## REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [5] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [6] Bo Li, Yan Zhang, and Xiquan Liang. Difference and difference quotient. *Formalized Mathematics*, 14(3):115–119, 2006.
- [7] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(1):17–21, 1992.
- [8] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [9] Andrzej Trybulec and Yatsuka Nakamura. On the decomposition of a simple closed curve into two arcs. *Formalized Mathematics*, 10(3):163–167, 2002.
- [10] Peng Wang and Bo Li. Several differentiation formulas of special functions. Part V. *Formalized Mathematics*, 15(3):73–79, 2007.
- [11] Renhong Wang. *Numerical approximation*. Higher Education Press, Beijing, 1999.
- [12] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [13] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(2):255–263, 1998.

Received October 25, 2007

---