

Solutions of Linear Equations

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Summary. In this paper I present the Kronecker-Capelli theorem which states that a system of linear equations has a solution if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix.

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The terminology and notation used in this paper are introduced in the following papers: [9], [24], [1], [2], [10], [25], [6], [8], [7], [3], [23], [21], [13], [5], [11], [12], [26], [15], [27], [19], [16], [22], [20], [28], [4], [17], [14], and [18].

1. PRELIMINARIES

For simplicity, we follow the rules: x denotes a set, i, j, k, l, m, n denote natural numbers, K denotes a field, N denotes a without zero finite subset of \mathbb{N} , a, b denote elements of K , $A, B, B_1, B_2, X, X_1, X_2$ denote matrices over K , A' denotes a matrix over K of dimension $m \times n$, B' denotes a matrix over K of dimension $m \times k$, and M denotes a square matrix over K of dimension n .

We now state a number of propositions:

- (1) If $\text{width } A = \text{len } B$, then $(a \cdot A) \cdot B = a \cdot (A \cdot B)$.
- (2) $\mathbf{1}_K \cdot A = A$ and $a \cdot (b \cdot A) = (a \cdot b) \cdot A$.
- (3) Let K be a non empty additive loop structure and f, g, h, w be finite sequences of elements of K . If $\text{len } f = \text{len } g$ and $\text{len } h = \text{len } w$, then $f \wedge h + g \wedge w = (f + g) \wedge (h + w)$.
- (4) Let K be a non empty multiplicative magma, f, g be finite sequences of elements of K , and a be an element of K . Then $a \cdot (f \wedge g) = (a \cdot f) \wedge (a \cdot g)$.

- (5) Let f be a function and p_1, p_2, f_1, f_2 be finite sequences. If $\text{rng } p_1 \subseteq \text{dom } f$ and $\text{rng } p_2 \subseteq \text{dom } f$ and $f_1 = f \cdot p_1$ and $f_2 = f \cdot p_2$, then $f \cdot (p_1 \hat{\ } p_2) = f_1 \hat{\ } f_2$.
- (6) Let f be a finite sequence of elements of \mathbb{N} and given n . Suppose f is one-to-one and $\text{rng } f \subseteq \text{Seg } n$ and for all i, j such that $i, j \in \text{dom } f$ and $i < j$ holds $f(i) < f(j)$. Then $\text{Sgm } \text{rng } f = f$.
- (7) Let K be an Abelian add-associative right zeroed right complementable non empty additive loop structure, p be a finite sequence of elements of K , and given i, j . Suppose $i, j \in \text{dom } p$ and $i \neq j$ and for every k such that $k \in \text{dom } p$ and $k \neq i$ and $k \neq j$ holds $p(k) = 0_K$. Then $\sum p = p_i + p_j$.
- (8) If $i \in \text{Seg } m$, then $(\text{Sgm}(\text{Seg}(n+m) \setminus \text{Seg } n))(i) = n + i$.
- (9) Let D be a non empty set, A be a matrix over D , and B_3, B_4, C_1, C_2 be without zero finite subsets of \mathbb{N} . Suppose $B_3 \times B_4 \subseteq$ the indices of A and $C_1 \times C_2 \subseteq$ the indices of A . Let B be a matrix over D of dimension $\text{card } B_3 \times \text{card } B_4$ and C be a matrix over D of dimension $\text{card } C_1 \times \text{card } C_2$. Suppose that for all natural numbers i, j, b_1, b_2, c_1, c_2 such that $\langle i, j \rangle \in (B_3 \times B_4) \cap (C_1 \times C_2)$ and $b_1 = (\text{Sgm } B_3)^{-1}(i)$ and $b_2 = (\text{Sgm } B_4)^{-1}(j)$ and $c_1 = (\text{Sgm } C_1)^{-1}(i)$ and $c_2 = (\text{Sgm } C_2)^{-1}(j)$ holds $B_{b_1, b_2} = C_{c_1, c_2}$. Then there exists a matrix M over D of dimension $\text{len } A \times \text{width } A$ such that $\text{Segm}(M, B_3, B_4) = B$ and $\text{Segm}(M, C_1, C_2) = C$ and for all i, j such that $\langle i, j \rangle \in (\text{the indices of } M) \setminus (B_3 \times B_4 \cup C_1 \times C_2)$ holds $M_{i, j} = A_{i, j}$.
- (10) Let P, Q, Q' be without zero finite subsets of \mathbb{N} . Suppose $P \times Q' \subseteq$ the indices of A . Let given i, j . Suppose $i \in \text{dom } A \setminus P$ and $j \in \text{Seg width } A \setminus Q$ and $A_{i, j} \neq 0_K$ and $Q \subseteq Q'$ and $\text{Line}(A, i) \cdot \text{Sgm } Q' = \text{card } Q' \mapsto 0_K$. Then $\text{rk}(A) > \text{rk}(\text{Segm}(A, P, Q))$.
- (11) For every N such that $N \subseteq \text{dom } A$ and for every i such that $i \in \text{dom } A \setminus N$ holds $\text{Line}(A, i) = \text{width } A \mapsto 0_K$ holds $\text{rk}(A) = \text{rk}(\text{Segm}(A, N, \text{Seg width } A))$.
- (12) For every N such that $N \subseteq \text{Seg width } A$ and for every i such that $i \in \text{Seg width } A \setminus N$ holds $A_{\square, i} = \text{len } A \mapsto 0_K$ holds $\text{rk}(A) = \text{rk}(\text{Segm}(A, \text{Seg len } A, N))$.
- (13) Let V be a vector space over K , U be a finite subset of V , u, v be vectors of V , and given a . If $u, v \in U$, then $\text{Lin}((U \setminus \{u\}) \cup \{u + a \cdot v\})$ is a subspace of $\text{Lin}(U)$.
- (14) Let V be a vector space over K , U be a finite subset of V , u, v be vectors of V , and given a . Suppose $u, v \in U$ and if $u = v$, then $a \neq -\mathbf{1}_K$ or $u = 0_V$. Then $\text{Lin}((U \setminus \{u\}) \cup \{u + a \cdot v\}) = \text{Lin}(U)$.

2. SELECTED PROPERTIES OF JOINING OPERATION OF TWO MATRICES

Let D be a non empty set, let n, m, k be natural numbers, let A be a matrix over D of dimension $n \times m$, and let B be a matrix over D of dimension $n \times k$. Then $A \frown B$ is a matrix over D of dimension $n \times (\text{width } A + \text{width } B)$.

We now state a number of propositions:

- (15) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i . If $i \in \text{Seg } n$, then $\text{Line}(A \frown B, i) = \text{Line}(A, i) \frown \text{Line}(B, i)$.
- (16) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i . If $i \in \text{Seg width } A$, then $(A \frown B)_{\square, i} = A_{\square, i}$.
- (17) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and given i . If $i \in \text{Seg width } B$, then $(A \frown B)_{\square, \text{width } A + i} = B_{\square, i}$.
- (18) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, B be a matrix over D of dimension $n \times k$, and p_3, p_4 be finite sequences of elements of D . If $\text{len } p_3 = \text{width } A$ and $\text{len } p_4 = \text{width } B$, then $\text{ReplaceLine}(A \frown B, i, p_3 \frown p_4) = (\text{ReplaceLine}(A, i, p_3)) \frown \text{ReplaceLine}(B, i, p_4)$.
- (19) Let D be a non empty set, A be a matrix over D of dimension $n \times m$, and B be a matrix over D of dimension $n \times k$. Then $\text{Segm}(A \frown B, \text{Seg } n, \text{Seg width } A) = A$ and $\text{Segm}(A \frown B, \text{Seg } n, \text{Seg}(\text{width } A + \text{width } B) \setminus \text{Seg width } A) = B$.
- (20) For all matrices A, B over K such that $\text{len } A = \text{len } B$ holds $\text{rk}(A) \leq \text{rk}(A \frown B)$ and $\text{rk}(B) \leq \text{rk}(A \frown B)$.
- (21) For all matrices A, B over K such that $\text{len } A = \text{len } B$ and $\text{len } A = \text{rk}(A)$ holds $\text{rk}(A) = \text{rk}(A \frown B)$.
- (22) For all matrices A, B over K such that $\text{len } A = \text{len } B$ and $\text{width } A = 0$ holds $A \frown B = B$ and $B \frown A = B$.
- (23) For all matrices A, B over K such that $B = 0_K^{(\text{len } A) \times m}$ holds $\text{rk}(A) = \text{rk}(A \frown B)$.
- (24) Let A, B be matrices over K . Suppose $\text{rk}(A) = \text{rk}(A \frown B)$ and $\text{len } A = \text{len } B$. Let given N . Suppose $N \subseteq \text{dom } A$ and for every i such that $i \in N$ holds $\text{Line}(A, i) = \text{width } A \mapsto 0_K$. Let given i . If $i \in N$, then $\text{Line}(B, i) = \text{width } B \mapsto 0_K$.

3. BASIC PROPERTIES OF TWO TRANSFORMATIONS WHICH TRANSFORM FINITE SEQUENCES TO MATRICES

For simplicity, we follow the rules: D is a non empty set, b_3 is a finite sequence of elements of D , b, f, g are finite sequences of elements of K , and M_1 is a matrix over D .

Let D be a non empty set and let b be a finite sequence of elements of D . The functor $\text{LineVec2Mx } b$ yielding a matrix over D of dimension $1 \times \text{len } b$ is defined by:

(Def. 1) $\text{LineVec2Mx } b = \langle b \rangle$.

The functor $\text{ColVec2Mx } b$ yielding a matrix over D of dimension $\text{len } b \times 1$ is defined by:

(Def. 2) $\text{ColVec2Mx } b = \langle b \rangle^T$.

One can prove the following propositions:

- (25) $M_1 = \text{LineVec2Mx } b_3$ iff $\text{Line}(M_1, 1) = b_3$ and $\text{len } M_1 = 1$.
- (26) If $\text{len } M_1 \neq 0$ or $\text{len } b_3 \neq 0$, then $M_1 = \text{ColVec2Mx } b_3$ iff $(M_1)_{\square, 1} = b_3$ and $\text{width } M_1 = 1$.
- (27) If $\text{len } f = \text{len } g$, then $\text{LineVec2Mx } f + \text{LineVec2Mx } g = \text{LineVec2Mx}(f + g)$.
- (28) If $\text{len } f = \text{len } g$, then $\text{ColVec2Mx } f + \text{ColVec2Mx } g = \text{ColVec2Mx}(f + g)$.
- (29) $a \cdot \text{LineVec2Mx } f = \text{LineVec2Mx}(a \cdot f)$.
- (30) $a \cdot \text{ColVec2Mx } f = \text{ColVec2Mx}(a \cdot f)$.
- (31) $\text{LineVec2Mx}(k \mapsto 0_K) = 0_K^{1 \times k}$.
- (32) $\text{ColVec2Mx}(k \mapsto 0_K) = 0_K^{k \times 1}$.

4. BASIS PROPERTIES OF THE SOLUTION OF LINEAR EQUATIONS

Let us consider K and let us consider A, B . The set of solutions of A and B is a set and is defined as follows:

(Def. 3) The set of solutions of A and $B = \{X : \text{len } X = \text{width } A \wedge \text{width } X = \text{width } B \wedge A \cdot X = B\}$.

We now state a number of propositions:

- (33) If the set of solutions of A and B is non empty, then $\text{len } A = \text{len } B$.
- (34) If $X \in$ the set of solutions of A and B and $i \in \text{Seg width } X$ and $X_{\square, i} = \text{len } X \mapsto 0_K$, then $B_{\square, i} = \text{len } B \mapsto 0_K$.
- (35) Suppose $X \in$ the set of solutions of A and B . Then $a \cdot X \in$ the set of solutions of A and $a \cdot B$ and $X \in$ the set of solutions of $a \cdot A$ and $a \cdot B$.
- (36) If $a \neq 0_K$, then the set of solutions of A and $B =$ the set of solutions of $a \cdot A$ and $a \cdot B$.

- (37) Suppose $X_1 \in$ the set of solutions of A and B_1 and $X_2 \in$ the set of solutions of A and B_2 and $\text{width } B_1 = \text{width } B_2$. Then $X_1 + X_2 \in$ the set of solutions of A and $B_1 + B_2$.
- (38) If $X \in$ the set of solutions of A' and B' , then $X \in$ the set of solutions of $\text{RLine}(A', i, a \cdot \text{Line}(A', i))$ and $\text{RLine}(B', i, a \cdot \text{Line}(B', i))$.
- (39) Suppose $X \in$ the set of solutions of A' and B' and $j \in \text{Seg } m$ and $i \neq j$. Then $X \in$ the set of solutions of $\text{RLine}(A', i, \text{Line}(A', i) + a \cdot \text{Line}(A', j))$ and $\text{RLine}(B', i, \text{Line}(B', i) + a \cdot \text{Line}(B', j))$.
- (40) Suppose $j \in \text{Seg } m$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then the set of solutions of A' and $B' =$ the set of solutions of $\text{RLine}(A', i, \text{Line}(A', i) + a \cdot \text{Line}(A', j))$ and $\text{RLine}(B', i, \text{Line}(B', i) + a \cdot \text{Line}(B', j))$.
- (41) If $X \in$ the set of solutions of A and B and $i \in \text{dom } A$ and $\text{Line}(A, i) = \text{width } A \mapsto 0_K$, then $\text{Line}(B, i) = \text{width } B \mapsto 0_K$.
- (42) Let n_1 be an element of \mathbb{N}^n . Suppose $\text{rng } n_1 \subseteq \text{dom } A$ and $n > 0$. Then the set of solutions of A and $B \subseteq$ the set of solutions of $\text{Segm}(A, n_1, \text{Sgm Seg width } A)$ and $\text{Segm}(B, n_1, \text{Sgm Seg width } B)$.
- (43) Let n_1 be an element of \mathbb{N}^n . Suppose $\text{rng } n_1 \subseteq \text{dom } A = \text{dom } B$ and $n > 0$ and for every i such that $i \in \text{dom } A \setminus \text{rng } n_1$ holds $\text{Line}(A, i) = \text{width } A \mapsto 0_K$ and $\text{Line}(B, i) = \text{width } B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of $\text{Segm}(A, n_1, \text{Sgm Seg width } A)$ and $\text{Segm}(B, n_1, \text{Sgm Seg width } B)$.
- (44) Let given N . Suppose $N \subseteq \text{dom } A$ and N is non empty. Then the set of solutions of A and $B \subseteq$ the set of solutions of $\text{Segm}(A, N, \text{Seg width } A)$ and $\text{Segm}(B, N, \text{Seg width } B)$.
- (45) Let given N . Suppose $N \subseteq \text{dom } A$ and N is non empty and $\text{dom } A = \text{dom } B$ and for every i such that $i \in \text{dom } A \setminus N$ holds $\text{Line}(A, i) = \text{width } A \mapsto 0_K$ and $\text{Line}(B, i) = \text{width } B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of $\text{Segm}(A, N, \text{Seg width } A)$ and $\text{Segm}(B, N, \text{Seg width } B)$.
- (46) Suppose $i \in \text{dom } A$ and $\text{len } A > 1$. Then the set of solutions of A and $B \subseteq$ the set of solutions of the deleting of i -row in A and the deleting of i -row in B .
- (47) Let given A, B, i . Suppose $i \in \text{dom } A$ and $\text{len } A > 1$ and $\text{Line}(A, i) = \text{width } A \mapsto 0_K$ and $i \in \text{dom } B$ and $\text{Line}(B, i) = \text{width } B \mapsto 0_K$. Then the set of solutions of A and $B =$ the set of solutions of the deleting of i -row in A and the deleting of i -row in B .
- (48) Let A be a matrix over K of dimension $n \times m$, B be a matrix over K of dimension $n \times k$, and P be a function from $\text{Seg } n$ into $\text{Seg } n$. Then
- (i) the set of solutions of A and $B \subseteq$ the set of solutions of $A \cdot P$ and $B \cdot P$, and

- (ii) if P is one-to-one, then the set of solutions of A and $B =$ the set of solutions of $A \cdot P$ and $B \cdot P$.
- (49) Let A be a matrix over K of dimension $n \times m$ and given N . Suppose $\text{card } N = n$ and $N \subseteq \text{Seg } m$ and $\text{Segm}(A, \text{Seg } n, N) = I_K^{n \times n}$ and $n > 0$. Then there exists a matrix M_2 over K of dimension $m - n \times m$ such that
- (i) $\text{Segm}(M_2, \text{Seg}(m - n), \text{Seg } m \setminus N) = I_K^{(m-n) \times (m-n)}$,
- (ii) $\text{Segm}(M_2, \text{Seg}(m - n), N) = -(\text{Segm}(A, \text{Seg } n, \text{Seg } m \setminus N))^T$, and
- (iii) for every l and for every matrix M over K of dimension $m \times l$ such that for every i such that $i \in \text{Seg } l$ holds there exists j such that $j \in \text{Seg}(m - n)$ and $M_{\square, i} = \text{Line}(M_2, j)$ or $M_{\square, i} = m \mapsto 0_K$ holds $M \in$ the set of solutions of A and $0_K^{n \times l}$.
- (50) Let A be a matrix over K of dimension $n \times m$, B be a matrix over K of dimension $n \times l$, and given N . Suppose $\text{card } N = n$ and $N \subseteq \text{Seg } m$ and $n > 0$ and $\text{Segm}(A, \text{Seg } n, N) = I_K^{n \times n}$. Then there exists a matrix X over K of dimension $m \times l$ such that $\text{Segm}(X, \text{Seg } m \setminus N, \text{Seg } l) = 0_K^{(m-n) \times l}$ and $\text{Segm}(X, N, \text{Seg } l) = B$ and $X \in$ the set of solutions of A and B .
- (51) Let A be a matrix over K of dimension $0 \times n$ and B be a matrix over K of dimension $0 \times m$. Then the set of solutions of A and $B = \{\emptyset\}$.
- (52) For every matrix B over K such that the set of solutions of $0_K^{n \times k}$ and B is non empty holds $B = 0_K^{n \times (\text{width } B)}$.
- (53) Let A be a matrix over K of dimension $n \times k$ and B be a matrix over K of dimension $n \times m$. Suppose $n > 0$. Suppose $x \in$ the set of solutions of A and B . Then x is a matrix over K of dimension $k \times m$.
- (54) Suppose $n > 0$ and $k > 0$. Then the set of solutions of $0_K^{n \times k}$ and $0_K^{n \times m} = \{X : X \text{ ranges over matrices over } K \text{ of dimension } k \times m\}$.
- (55) If $n > 0$ and the set of solutions of $0_K^{n \times 0}$ and $0_K^{n \times m}$ is non empty, then $m = 0$.
- (56) The set of solutions of $0_K^{n \times 0}$ and $0_K^{n \times 0} = \{\emptyset\}$.

5. GAUSSIAN ELIMINATIONS

In this article we present several logical schemes. The scheme *GAUSS1* deals with a field \mathcal{A} , natural numbers \mathcal{B} , \mathcal{C} , \mathcal{D} , a matrix \mathcal{E} over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$, a matrix \mathcal{F} over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$, a 4-ary functor \mathcal{F} yielding a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$, and a binary predicate \mathcal{P} , and states that:

There exists a matrix A' over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$ and there exists a matrix B' over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$ and there exists a without zero finite subset N of \mathbb{N} such that

$N \subseteq \text{Seg } \mathcal{C}$ and $\text{rk}(\mathcal{E}) = \text{rk}(A')$ and $\text{rk}(\mathcal{E}) = \text{card } N$ and $\mathcal{P}[A', B']$ and $\text{Segm}(A', \text{Seg } \text{card } N, N)$ is diagonal and for every i

such that $i \in \text{Seg card } N$ holds $A'_{i,(\text{Sgm } N)_i} \neq 0_{\mathcal{A}}$ and for every i such that $i \in \text{dom } A'$ and $i > \text{card } N$ holds $\text{Line}(A', i) = \mathcal{C} \mapsto 0_{\mathcal{A}}$ and for all i, j such that $i \in \text{Seg card } N$ and $j \in \text{Seg width } A'$ and $j < (\text{Sgm } N)(i)$ holds $A'_{i,j} = 0_{\mathcal{A}}$

provided the parameters meet the following requirements:

- $\mathcal{P}[\mathcal{E}, \mathcal{F}]$, and
- Let A' be a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$ and B' be a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$. Suppose $\mathcal{P}[A', B']$. Let given i, j . Suppose $i \neq j$ and $j \in \text{dom } A'$. Let a be an element of \mathcal{A} . Then $\mathcal{P}[\text{RLine}(A', i, \text{Line}(A', i) + a \cdot \text{Line}(A', j)), \mathcal{F}(B', i, j, a)]$.

The scheme *GAUSS2* deals with a field \mathcal{A} , natural numbers $\mathcal{B}, \mathcal{C}, \mathcal{D}$, a matrix \mathcal{E} over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$, a matrix \mathcal{F} over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$, a 4-ary functor \mathcal{F} yielding a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$, and a binary predicate \mathcal{P} , and states that:

There exists a matrix A' over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$ and there exists a matrix B' over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$ and there exists a without zero finite subset N of \mathbb{N} such that

$N \subseteq \text{Seg } \mathcal{C}$ and $\text{rk}(\mathcal{E}) = \text{rk}(A')$ and $\text{rk}(\mathcal{E}) = \text{card } N$ and $\mathcal{P}[A', B']$ and $\text{Segm}(A', \text{Seg card } N, N) = I_{\mathcal{A}}^{\text{card } N \times \text{card } N}$ and for every i such that $i \in \text{dom } A'$ and $i > \text{card } N$ holds $\text{Line}(A', i) = \mathcal{C} \mapsto 0_{\mathcal{A}}$ and for all i, j such that $i \in \text{Seg card } N$ and $j \in \text{Seg width } A'$ and $j < (\text{Sgm } N)(i)$ holds $A'_{i,j} = 0_{\mathcal{A}}$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\mathcal{E}, \mathcal{F}]$, and
- Let A' be a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{C}$ and B' be a matrix over \mathcal{A} of dimension $\mathcal{B} \times \mathcal{D}$. Suppose $\mathcal{P}[A', B']$. Let a be an element of \mathcal{A} and given i, j . If $j \in \text{dom } A'$ and if $i = j$, then $a \neq -\mathbf{1}_{\mathcal{A}}$, then $\mathcal{P}[\text{RLine}(A', i, \text{Line}(A', i) + a \cdot \text{Line}(A', j)), \mathcal{F}(B', i, j, a)]$.

6. THE MAIN THEOREM

We now state the proposition

- (57) Let A, B be matrices over K . Suppose $\text{len } A = \text{len } B$ and if $\text{width } A = 0$, then $\text{width } B = 0$. Then $\text{rk}(A) = \text{rk}(A \cap B)$ if and only if the set of solutions of A and B is non empty.

7. SPACE OF SOLUTIONS OF LINEAR EQUATIONS

Let us consider K , let A be a matrix over K , and let b be a finite sequence of elements of K . The set of solutions of A and b is defined by:

(Def. 4) The set of solutions of A and $b = \{f : \text{ColVec2Mx } f \in \text{the set of solutions of } A \text{ and } \text{ColVec2Mx } b\}$.

We now state two propositions:

- (58) For every x such that $x \in \text{the set of solutions of } A \text{ and } \text{ColVec2Mx } b$ there exists f such that $x = \text{ColVec2Mx } f$ and $\text{len } f = \text{width } A$.
- (59) For every f such that $\text{ColVec2Mx } f \in \text{the set of solutions of } A \text{ and } \text{ColVec2Mx } b$ holds $\text{len } f = \text{width } A$.

Let us consider K , let A be a matrix over K , and let b be a finite sequence of elements of K . Then the set of solutions of A and b is a subset of the width A -dimension vector space over K .

Let us consider K , let A be a matrix over K , and let k be an element of \mathbb{N} . Note that the set of solutions of A and $k \mapsto 0_K$ is linearly closed.

We now state two propositions:

- (60) If the set of solutions of A and b is non empty and $\text{width } A = 0$, then $\text{len } A = 0$.
- (61) If $\text{width } A \neq 0$ or $\text{len } A = 0$, then the set of solutions of A and $\text{len } A \mapsto 0_K$ is non empty.

Let us consider K and let A be a matrix over K . Let us assume that if $\text{width } A = 0$, then $\text{len } A = 0$. The space of solutions of A is a strict subspace of the width A -dimension vector space over K and is defined by:

(Def. 5) The carrier of the space of solutions of $A = \text{the set of solutions of } A \text{ and } \text{len } A \mapsto 0_K$.

The following propositions are true:

- (62) Let A be a matrix over K and b be a finite sequence of elements of K . Suppose the set of solutions of A and b is non empty. Then the set of solutions of A and b is a coset of the space of solutions of A .
- (63) Let given A . Suppose if $\text{width } A = 0$, then $\text{len } A = 0$ and $\text{rk}(A) = 0$. Then the space of solutions of $A = \text{the width } A\text{-dimension vector space over } K$.
- (64) For every A such that the space of solutions of $A = \text{the width } A\text{-dimension vector space over } K$ holds $\text{rk}(A) = 0$.
- (65) Let given i, j . Suppose $j \in \text{Seg } m$ and $n > 0$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then the space of solutions of $A' = \text{the space of solutions of } \text{RLine}(A', i, \text{Line}(A', i) + a \cdot \text{Line}(A', j))$.
- (66) Let given N . Suppose $N \subseteq \text{dom } A$ and N is non empty and $\text{width } A > 0$ and for every i such that $i \in \text{dom } A \setminus N$ holds $\text{Line}(A, i) = \text{width } A \mapsto 0_K$. Then the space of solutions of $A = \text{the space of solutions of } \text{Segm}(A, N, \text{Seg width } A)$.
- (67) Let A be a matrix over K of dimension $n \times m$ and given N . Suppose $\text{card } N = n$ and $N \subseteq \text{Seg } m$ and $\text{Segm}(A, \text{Seg } n, N) = I_K^{n \times n}$ and $n > 0$

- and $m -' n > 0$. Then there exists a matrix M_2 over K of dimension $m -' n \times m$ such that $\text{Segm}(M_2, \text{Seg}(m -' n), \text{Seg } m \setminus N) = I_K^{(m -' n) \times (m -' n)}$ and $\text{Segm}(M_2, \text{Seg}(m -' n), N) = -(\text{Segm}(A, \text{Seg } n, \text{Seg } m \setminus N))^T$ and $\text{Lin}(\text{lines}(M_2)) =$ the space of solutions of A .
- (68) For every A such that if $\text{width } A = 0$, then $\text{len } A = 0$ holds $\dim(\text{the space of solutions of } A) = \text{width } A - \text{rk}(A)$.
- (69) Let M be a matrix over K of dimension $n \times m$ and given i, j, a . Suppose M is without repeated line and $j \in \text{dom } M$ and if $i = j$, then $a \neq -\mathbf{1}_K$. Then $\text{Lin}(\text{lines}(M)) = \text{Lin}(\text{lines}(\text{RLine}(M, i, \text{Line}(M, i) + a \cdot \text{Line}(M, j))))$.
- (70) Let W be a subspace of the m -dimension vector space over K . Then there exists a matrix A over K of dimension $\dim(W) \times m$ and there exists a without zero finite subset N of \mathbb{N} such that $N \subseteq \text{Seg } m$ and $\dim(W) = \text{card } N$ and $\text{Segm}(A, \text{Seg } \dim(W), N) = I_K^{\dim(W) \times \dim(W)}$ and $\text{rk}(A) = \dim(W)$ and $\text{lines}(A)$ is a basis of W .
- (71) Let W be a strict subspace of the m -dimension vector space over K . Suppose $\dim(W) < m$. Then there exists a matrix A over K of dimension $m -' \dim(W) \times m$ and there exists a without zero finite subset N of \mathbb{N} such that $\text{card } N = m -' \dim(W)$ and $N \subseteq \text{Seg } m$ and $\text{Segm}(A, \text{Seg}(m -' \dim(W)), N) = I_K^{(m -' \dim(W)) \times (m -' \dim(W))}$ and $W =$ the space of solutions of A .
- (72) Let A, B be matrices over K . Suppose $\text{width } A = \text{len } B$ and if $\text{width } A = 0$, then $\text{len } A = 0$ and if $\text{width } B = 0$, then $\text{len } B = 0$. Then the space of solutions of B is a subspace of the space of solutions of $A \cdot B$.
- (73) For all matrices A, B over K such that $\text{width } A = \text{len } B$ holds $\text{rk}(A \cdot B) \leq \text{rk}(A)$ and $\text{rk}(A \cdot B) \leq \text{rk}(B)$.
- (74) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K . Suppose $\text{Det } A \neq 0_K$ and $\text{width } A = \text{len } B$ and if $\text{width } B = 0$, then $\text{len } B = 0$. Then the space of solutions of $B =$ the space of solutions of $A \cdot B$.
- (75) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K . If $\text{width } A = \text{len } B$ and $\text{Det } A \neq 0_K$, then $\text{rk}(A \cdot B) = \text{rk}(B)$.
- (76) Let A be a matrix over K of dimension $n \times n$ and B be a matrix over K . If $\text{len } A = \text{width } B$ and $\text{Det } A \neq 0_K$, then $\text{rk}(B \cdot A) = \text{rk}(B)$.

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