

The Real Vector Spaces of Finite Sequences are Finite Dimensional

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Summary. In this paper we show the finite dimensionality of real linear spaces with their carriers equal \mathcal{R}^n . We also give the standard basis of such spaces. For the set \mathcal{R}^n we introduce the concepts of linear manifold subsets and orthogonal subsets. The cardinality of orthonormal basis of discussed spaces is proved to equal n .

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The articles [32], [7], [11], [33], [9], [2], [8], [5], [31], [4], [6], [18], [13], [22], [20], [14], [1], [21], [29], [28], [26], [3], [23], [10], [12], [30], [19], [34], [16], [17], [25], [15], [24], and [27] provide the notation and terminology for this paper.

1. PRELIMINARIES

We use the following convention: i, j, n are elements of \mathbb{N} , z, B_0 are sets, and f, x_0 are real-valued finite sequences.

Next we state several propositions:

- (1) For all functions f, g holds $\text{dom}(f \cdot g) = \text{dom } g \cap g^{-1}(\text{dom } f)$.
- (2) For every binary relation R and for every set Y such that $\text{rng } R \subseteq Y$ holds $R^{-1}(Y) = \text{dom } R$.

(3) Let X be a set, Y be a non empty set, and f be a function from X into Y . If f is bijective, then $\overline{X} = \overline{Y}$.

(4) $\langle z \rangle \cdot \langle 1 \rangle = \langle z \rangle$.

(5) For every element x of \mathcal{R}^0 holds $x = \varepsilon_{\mathbb{R}}$.

(6) For all elements a, b, c of \mathcal{R}^n holds $(a - b) + c + b = a + c$.

Let f_1, f_2 be finite sequences. One can verify that $\langle f_1, f_2 \rangle$ is finite sequence-like.

Let D be a set and let f_1, f_2 be finite sequences of elements of D . Then $\langle f_1, f_2 \rangle$ is a finite sequence of elements of $D \times D$.

Let h be a real-valued finite sequence. Let us observe that h is increasing if and only if:

(Def. 1) For every i such that $1 \leq i < \text{len } h$ holds $h(i) < h(i + 1)$.

One can prove the following four propositions:

(7) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j . If $i < j$ and $1 \leq i$ and $j \leq \text{len } h$, then $h(i) < h(j)$.

(8) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j . If $i \leq j$ and $1 \leq i$ and $j \leq \text{len } h$, then $h(i) \leq h(j)$.

(9) Let h be a natural-valued finite sequence. Suppose h is increasing. Let given i . If $1 \leq i \leq \text{len } h$ and $1 \leq h(1)$, then $i \leq h(i)$.

(10) Let V be a real linear space and X be a subspace of V . Suppose V is strict and X is strict and the carrier of $X =$ the carrier of V . Then $X = V$.

Let D be a set, let F be a finite sequence of elements of D , and let h be a permutation of $\text{dom } F$. The functor $F \circ h$ yields a finite sequence of elements of D and is defined as follows:

(Def. 2) $F \circ h = F \cdot h$.

One can prove the following propositions:

(11) Let D be a non empty set and f be a finite sequence of elements of D . If $1 \leq i \leq \text{len } f$ and $1 \leq j \leq \text{len } f$, then $(\text{Swap}(f, i, j))(i) = f(j)$ and $(\text{Swap}(f, i, j))(j) = f(i)$.

(12) \emptyset is a permutation of \emptyset .

(13) $\langle 1 \rangle$ is a permutation of $\{1\}$.

(14) For every finite sequence h of elements of \mathbb{R} holds h is one-to-one iff $\text{sort}_a h$ is one-to-one.

(15) Let h be a finite sequence of elements of \mathbb{N} . Suppose h is one-to-one. Then there exists a permutation h_3 of $\text{dom } h$ and there exists a finite sequence h_2 of elements of \mathbb{N} such that $h_2 = h \cdot h_3$ and h_2 is increasing and $\text{dom } h = \text{dom } h_2$ and $\text{rng } h = \text{rng } h_2$.

2. ORTHOGONAL BASIS

Let B_0 be a set. We say that B_0 is \mathbb{R} -orthogonal if and only if:

- (Def. 3) For all real-valued finite sequences x, y such that $x, y \in B_0$ and $x \neq y$ holds $|(x, y)| = 0$.

Let us observe that every set which is empty is also \mathbb{R} -orthogonal.

We now state the proposition

- (16) B_0 is \mathbb{R} -orthogonal if and only if for all points x, y of \mathcal{E}_T^n such that $x, y \in B_0$ and $x \neq y$ holds x, y are orthogonal.

Let B_0 be a set. We say that B_0 is \mathbb{R} -normal if and only if:

- (Def. 4) For every real-valued finite sequence x such that $x \in B_0$ holds $|x| = 1$.

Let us observe that every set which is empty is also \mathbb{R} -normal.

Let us observe that there exists a set which is \mathbb{R} -normal.

Let B_0, B_1 be \mathbb{R} -normal sets. One can verify that $B_0 \cup B_1$ is \mathbb{R} -normal.

One can prove the following propositions:

- (17) If $|f| = 1$, then $\{f\}$ is \mathbb{R} -normal.
 (18) If B_0 is \mathbb{R} -normal and $|x_0| = 1$, then $B_0 \cup \{x_0\}$ is \mathbb{R} -normal.

Let B_0 be a set. We say that B_0 is \mathbb{R} -orthonormal if and only if:

- (Def. 5) B_0 is \mathbb{R} -orthogonal and \mathbb{R} -normal.

Let us note that every set which is \mathbb{R} -orthonormal is also \mathbb{R} -orthogonal and \mathbb{R} -normal and every set which is \mathbb{R} -orthogonal and \mathbb{R} -normal is also \mathbb{R} -orthonormal.

Let us observe that $\{\langle 1 \rangle\}$ is \mathbb{R} -orthonormal.

Let us observe that there exists a set which is \mathbb{R} -orthonormal and non empty.

Let us consider n . One can verify that there exists a subset of \mathcal{R}^n which is \mathbb{R} -orthonormal.

Let us consider n and let B_0 be a subset of \mathcal{R}^n . We say that B_0 is complete if and only if:

- (Def. 6) For every \mathbb{R} -orthonormal subset B of \mathcal{R}^n such that $B_0 \subseteq B$ holds $B = B_0$.

Let n be an element of \mathbb{N} and let B_0 be a subset of \mathcal{R}^n . We say that B_0 is orthogonal basis if and only if:

- (Def. 7) B_0 is \mathbb{R} -orthonormal and complete.

Let us consider n . One can verify that every subset of \mathcal{R}^n which is orthogonal basis is also \mathbb{R} -orthonormal and complete and every subset of \mathcal{R}^n which is \mathbb{R} -orthonormal and complete is also orthogonal basis.

The following propositions are true:

- (19) For every subset B_0 of \mathcal{R}^0 such that B_0 is orthogonal basis holds $B_0 = \emptyset$.

- (20) Let B_0 be a subset of \mathcal{R}^n and y be an element of \mathcal{R}^n . Suppose B_0 is orthogonal basis and for every element x of \mathcal{R}^n such that $x \in B_0$ holds $|(x, y)| = 0$. Then $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$.

3. LINEAR MANIFOLDS

Let us consider n and let X be a subset of \mathcal{R}^n . We say that X is linear manifold if and only if:

- (Def. 8) For all elements x, y of \mathcal{R}^n and for all elements a, b of \mathbb{R} such that $x, y \in X$ holds $a \cdot x + b \cdot y \in X$.

Let us consider n . Observe that $\Omega_{\mathcal{R}^n}$ is linear manifold.

The following proposition is true

- (21) $\{\underbrace{\langle 0, \dots, 0 \rangle}_n\}$ is linear manifold.

Let us consider n . Observe that $\{\underbrace{\langle 0, \dots, 0 \rangle}_n\}$ is linear manifold.

Let us consider n and let X be a subset of \mathcal{R}^n . The linear span of X yielding a subset of \mathcal{R}^n is defined by:

- (Def. 9) The linear span of $X = \bigcap \{Y \subseteq \mathcal{R}^n : Y \text{ is linear manifold} \wedge X \subseteq Y\}$.

Let us consider n and let X be a subset of \mathcal{R}^n . Observe that the linear span of X is linear manifold.

Let us consider n and let f be a finite sequence of elements of \mathcal{R}^n . The functor $\sum f$ yielding an element of \mathcal{R}^n is defined as follows:

- (Def. 10)(i) There exists a finite sequence g of elements of \mathcal{R}^n such that $\text{len } f = \text{len } g$ and $f(1) = g(1)$ and for every natural number i such that $1 \leq i < \text{len } f$ holds $g(i+1) = g_i + f_{i+1}$ and $\sum f = g(\text{len } f)$ if $\text{len } f > 0$,
(ii) $\sum f = \underbrace{\langle 0, \dots, 0 \rangle}_n$, otherwise.

Let n be a natural number and let f be a finite sequence of elements of \mathcal{R}^n . The functor $\text{accum } f$ yields a finite sequence of elements of \mathcal{R}^n and is defined as follows:

- (Def. 11) $\text{len } f = \text{len accum } f$ and $f(1) = (\text{accum } f)(1)$ and for every natural number i such that $1 \leq i < \text{len } f$ holds $(\text{accum } f)(i+1) = (\text{accum } f)_i + f_{i+1}$.

We now state several propositions:

- (22) For every finite sequence f of elements of \mathcal{R}^n such that $\text{len } f > 0$ holds $(\text{accum } f)(\text{len } f) = \sum f$.
(23) For all finite sequences F, F_2 of elements of \mathcal{R}^n and for every permutation h of $\text{dom } F$ such that $F_2 = F \circ h$ holds $\sum F_2 = \sum F$.
(24) For every element k of \mathbb{N} holds $\sum k \mapsto \underbrace{\langle 0, \dots, 0 \rangle}_n = \underbrace{\langle 0, \dots, 0 \rangle}_n$.

- (25) Let g be a finite sequence of elements of \mathcal{R}^n , h be a finite sequence of elements of \mathbb{N} , and F be a finite sequence of elements of \mathcal{R}^n . Suppose h is increasing and $\text{rng } h \subseteq \text{dom } g$ and $F = g \cdot h$ and for every element i of \mathbb{N} such that $i \in \text{dom } g$ and $i \notin \text{rng } h$ holds $g(i) = \underbrace{\langle 0, \dots, 0 \rangle}_n$. Then
- $$\sum g = \sum F.$$
- (26) Let g be a finite sequence of elements of \mathcal{R}^n , h be a finite sequence of elements of \mathbb{N} , and F be a finite sequence of elements of \mathcal{R}^n . Suppose h is one-to-one and $\text{rng } h \subseteq \text{dom } g$ and $F = g \cdot h$ and for every element i of \mathbb{N} such that $i \in \text{dom } g$ and $i \notin \text{rng } h$ holds $g(i) = \underbrace{\langle 0, \dots, 0 \rangle}_n$. Then
- $$\sum g = \sum F.$$

4. STANDARD BASIS

Let us consider n, i . Then the base finite sequence of n and i is an element of \mathcal{R}^n .

The following propositions are true:

- (27) Let i_1, i_2 be elements of \mathbb{N} . Suppose that
- (i) $1 \leq i_1$,
 - (ii) $i_1 \leq n$,
 - (iii) $1 \leq i_2$,
 - (iv) $i_2 \leq n$, and
 - (v) the base finite sequence of n and $i_1 =$ the base finite sequence of n and i_2 .
- Then $i_1 = i_2$.
- (28) $^2(\text{the base finite sequence of } n \text{ and } i) = \text{the base finite sequence of } n \text{ and } i$.
- (29) If $1 \leq i \leq n$, then $\sum \text{the base finite sequence of } n \text{ and } i = 1$.
- (30) If $1 \leq i \leq n$, then $|\text{the base finite sequence of } n \text{ and } i| = 1$.
- (31) Suppose $1 \leq i \leq n$ and $1 \leq j \leq n$ and $i \neq j$. Then $|(\text{the base finite sequence of } n \text{ and } i, \text{ the base finite sequence of } n \text{ and } j)| = 0$.
- (32) For every element x of \mathcal{R}^n such that $1 \leq i \leq n$ holds $|(x, \text{the base finite sequence of } n \text{ and } i)| = x(i)$.

Let us consider n and let x_0 be an element of \mathcal{R}^n . The functor $\text{ProjFinSeq } x_0$ yields a finite sequence of elements of \mathcal{R}^n and is defined by the conditions (Def. 12).

- (Def. 12)(i) $\text{len ProjFinSeq } x_0 = n$, and
- (ii) for every i such that $1 \leq i \leq n$ holds $(\text{ProjFinSeq } x_0)(i) = |(x_0, \text{the base finite sequence of } n \text{ and } i)| \cdot \text{the base finite sequence of } n \text{ and } i$.

The following proposition is true

(33) For every element x_0 of \mathcal{R}^n holds $x_0 = \sum \text{ProjFinSeq } x_0$.

Let us consider n . The functor $\mathbb{R}\text{N-Base } n$ yields a subset of \mathcal{R}^n and is defined by:

(Def. 13) $\mathbb{R}\text{N-Base } n = \{\text{the base finite sequence of } n \text{ and } i; i \text{ ranges over elements of } \mathbb{N}: 1 \leq i \wedge i \leq n\}$.

Next we state the proposition

(34) For every non zero element n of \mathbb{N} holds $\mathbb{R}\text{N-Base } n \neq \emptyset$.

Let us mention that $\mathbb{R}\text{N-Base } 0$ is empty.

Let n be a non zero element of \mathbb{N} . Note that $\mathbb{R}\text{N-Base } n$ is non empty.

Let us consider n . Observe that $\mathbb{R}\text{N-Base } n$ is orthogonal basis.

Let us consider n . Observe that there exists a subset of \mathcal{R}^n which is orthogonal basis.

Let us consider n . An orthogonal basis of n is an orthogonal basis subset of \mathcal{R}^n .

Let n be a non zero element of \mathbb{N} . Observe that every orthogonal basis of n is non empty.

5. FINITE REAL UNITARY SPACES AND FINITE REAL LINEAR SPACES

Let n be an element of \mathbb{N} . Observe that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is constituted finite sequences. Let n be an element of \mathbb{N} . One can check that every element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is real-valued.

Let n be an element of \mathbb{N} , let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can verify that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let n be an element of \mathbb{N} , let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, let y be a real-valued function, and let a, b be elements of \mathbb{R} . Observe that $a \cdot x$ and by can be identified when $a = b$ and $x = y$.

Let n be an element of \mathbb{N} , let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a be a real-valued function. Observe that $-x$ and $-a$ can be identified when $x = a$.

Let n be an element of \mathbb{N} , let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can check that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$. The following three propositions are true:

(35) Let n be an element of \mathbb{N} , x, y be elements of \mathcal{R}^n , and u, v be points of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $x = u$ and $y = v$, then $\otimes_{\mathcal{E}^n}(\langle u, v \rangle) = |(x, y)|$.

(36) Let n, j be elements of \mathbb{N} , F be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, B_2 be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, v_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and l be a linear combination of B_2 . Suppose F is one-to-one and B_2 is \mathbb{R} -orthogonal and $\text{rng } F = \text{the support of } l$ and $v_0 \in B_2$ and $j \in \text{dom}(lF)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle v_0, \sum lF \rangle) = \otimes_{\mathcal{E}^n}(\langle v_0, l(F_j) \cdot v_0 \rangle)$.

- (37) Let n be an element of \mathbb{N} , f be a finite sequence of elements of \mathcal{R}^n , and g be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $f = g$, then $\sum f = \sum g$.

Let A be a set. Note that $\mathbb{R}_{\mathbb{R}}^A$ is constituted functions.

Let us consider n . Observe that $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ is constituted finite sequences.

Let A be a set. One can verify that every element of $\mathbb{R}_{\mathbb{R}}^A$ is real-valued.

Let A be a set, let x, y be vectors of $\mathbb{R}_{\mathbb{R}}^A$, and let a, b be real-valued functions. Observe that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let A be a set, let x be a vector of $\mathbb{R}_{\mathbb{R}}^A$, let y be a real-valued function, and let a, b be elements of \mathbb{R} . Observe that $a \cdot x$ and $b y$ can be identified when $a = b$ and $x = y$.

Let A be a set, let x be a vector of $\mathbb{R}_{\mathbb{R}}^A$, and let a be a real-valued function. One can check that $-x$ and $-a$ can be identified when $x = a$.

Let A be a set, let x, y be vectors of $\mathbb{R}_{\mathbb{R}}^A$, and let a, b be real-valued functions. Observe that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

The following propositions are true:

- (38) Let X be a subspace of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, x be an element of \mathcal{R}^n , and a be a real number. If $x \in$ the carrier of X , then $a \cdot x \in$ the carrier of X .
- (39) Let X be a subspace of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ and x, y be elements of \mathcal{R}^n . Suppose $x \in$ the carrier of X and $y \in$ the carrier of X . Then $x + y \in$ the carrier of X .
- (40) Let X be a subspace of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, x, y be elements of \mathcal{R}^n , and a, b be real numbers. Suppose $x \in$ the carrier of X and $y \in$ the carrier of X . Then $a \cdot x + b \cdot y \in$ the carrier of X .
- (41) For all elements x, y of \mathcal{R}^n and for all points u, v of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ such that $x = u$ and $y = v$ holds $\otimes_{\mathcal{E}^n}(\langle u, v \rangle) = |(x, y)|$.
- (42) Let F be a finite sequence of elements of the carrier of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, B_2 be a subset of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, v_0 be an element of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, and l be a linear combination of B_2 . Suppose F is one-to-one and B_2 is \mathbb{R} -orthogonal and $\text{rng } F =$ the support of l and $v_0 \in B_2$ and $j \in \text{dom}(l F)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle v_0, \sum l F \rangle) = \otimes_{\mathcal{E}^n}(\langle v_0, l(F_j) \cdot v_0 \rangle)$.

Let us consider n . Note that every subset of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ which is \mathbb{R} -orthonormal is also linearly independent.

Let n be an element of \mathbb{N} . Note that every subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ which is \mathbb{R} -orthonormal is also linearly independent. Next we state the proposition

- (43) Let B_2 be a subset of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, x, y be elements of \mathcal{R}^n , and a be a real number. If B_2 is linearly independent and $x, y \in B_2$ and $y = a \cdot x$, then $x = y$.

6. FINITE DIMENSIONALITY OF THE SPACES

Let us consider n . One can check that \mathbb{RN} -Base n is finite.

The following propositions are true:

- (44) $\text{card } \mathbb{RN}\text{-Base } n = n$.
- (45) Let f be a finite sequence of elements of \mathcal{R}^n and g be a finite sequence of elements of the carrier of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$. If $f = g$, then $\sum f = \sum g$.
- (46) Let x_0 be an element of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ and B be a subset of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$. If $B = \mathbb{RN}\text{-Base } n$, then there exists a linear combination l of B such that $x_0 = \sum l$.
- (47) Let n be an element of \mathbb{N} , x_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and B be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $B = \mathbb{RN}\text{-Base } n$, then there exists a linear combination l of B such that $x_0 = \sum l$.
- (48) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ such that $B = \mathbb{RN}\text{-Base } n$ holds B is a basis of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$.

Let us consider n . Observe that $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ is finite dimensional.

We now state several propositions:

- (49) $\dim(\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}) = n$.
- (50) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ such that B is a basis of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ holds $\overline{B} = n$.
- (51) \emptyset is a basis of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } 0}$.
- (52) For every element n of \mathbb{N} holds $\mathbb{RN}\text{-Base } n$ is a basis of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$.
- (53) Every orthogonal basis of n is a basis of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$.

Let n be an element of \mathbb{N} . Note that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is finite dimensional.

We now state two propositions:

- (54) For every element n of \mathbb{N} holds $\dim(\langle \mathcal{E}^n, (\cdot|\cdot) \rangle) = n$.
- (55) For every orthogonal basis B of n holds $\overline{B} = n$.

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