FORMALIZED MATHEMATICS
Vol. 17, No. 1, Pages 1–9, 2009
DOI: 10.2478/v10037-009-0001-2

The Real Vector Spaces of Finite Sequences are Finite Dimensional

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Summary. In this paper we show the finite dimensionality of real linear spaces with their carriers equal $\mathcal{R}^n$. We also give the standard basis of such spaces. For the set $\mathcal{R}^n$ we introduce the concepts of linear manifold subsets and orthogonal subsets. The cardinality of orthonormal basis of discussed spaces is proved to equal n.

MML identifier: `EUCLID_7`, version: `7.11.01 4.117.1046`

The articles [32], [7], [11], [33], [9], [2], [8], [5], [31], [4], [6], [18], [13], [22], [20], [14], [1], [21], [29], [28], [26], [3], [23], [10], [12], [30], [19], [34], [16], [17], [25], [15], [24], and [27] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: i, j, n are elements of $\mathbb{N}$, z, B_0 are sets, and f, x_0 are real-valued finite sequences.

Next we state several propositions:

(1) For all functions f, g holds $\mathrm{dom}(f \cdot g) = \mathrm{dom}\, g \cap g^{-1}(\mathrm{dom}\, f)$.

(2) For every binary relation R and for every set Y such that $\mathrm{rng}\, R \subseteq Y$ holds $R^{-1}(Y) = \mathrm{dom}\, R$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Let X be a set, Y be a non empty set, and f be a function from X into Y. If f is bijective, then $\overline{\overline{X}} = \overline{\overline{Y}}$.

(4) $\langle z \rangle \cdot \langle 1 \rangle = \langle z \rangle$.

(5) For every element x of $\mathcal{R}^0$ holds $x = \varepsilon_{\mathbb{R}}$.

(6) For all elements a, b, c of $\mathcal{R}^n$ holds $(a - b) + c + b = a + c$.

Let f_1, f_2 be finite sequences. One can verify that $\langle f_1, f_2 \rangle$ is finite sequence-like.

Let D be a set and let f_1, f_2 be finite sequences of elements of D. Then $\langle f_1, f_2 \rangle$ is a finite sequence of elements of $D \times D$.

Let h be a real-valued finite sequence. Let us observe that h is increasing if and only if:

(Def. 1) For every i such that $1 \leq i < \operatorname{len} h$ holds $h(i) < h(i+1)$.

One can prove the following four propositions:

(7) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j. If $i < j$ and $1 \leq i$ and $j \leq \operatorname{len} h$, then $h(i) < h(j)$.

(8) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j. If $i \leq j$ and $1 \leq i$ and $j \leq \operatorname{len} h$, then $h(i) \leq h(j)$.

(9) Let h be a natural-valued finite sequence. Suppose h is increasing. Let given i. If $1 \leq i \leq \operatorname{len} h$ and $1 \leq h(1)$, then $i \leq h(i)$.

(10) Let V be a real linear space and X be a subspace of V. Suppose V is strict and X is strict and the carrier of $X =$ the carrier of V. Then $X = V$.

Let D be a set, let F be a finite sequence of elements of D, and let h be a permutation of $\operatorname{dom} F$. The functor $F \circ h$ yields a finite sequence of elements of D and is defined as follows:

(Def. 2) $F \circ h = F \cdot h$.

One can prove the following propositions:

(11) Let D be a non empty set and f be a finite sequence of elements of D. If $1 \leq i \leq \operatorname{len} f$ and $1 \leq j \leq \operatorname{len} f$, then $(\operatorname{Swap}(f, i, j))(i) = f(j)$ and $(\operatorname{Swap}(f, i, j))(j) = f(i)$.

(12) $\emptyset$ is a permutation of $\emptyset$.

(13) $\langle 1 \rangle$ is a permutation of $\{1\}$.

(14) For every finite sequence h of elements of $\mathbb{R}$ holds h is one-to-one iff $\operatorname{sort_a} h$ is one-to-one.

(15) Let h be a finite sequence of elements of $\mathbb{N}$. Suppose h is one-to-one. Then there exists a permutation h_3 of $\operatorname{dom} h$ and there exists a finite sequence h_2 of elements of $\mathbb{N}$ such that $h_2 = h \cdot h_3$ and h_2 is increasing and $\operatorname{dom} h = \operatorname{dom} h_2$ and $\operatorname{rng} h = \operatorname{rng} h_2$.

2. Orthogonal Basis

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-orthogonal if and only if:

(Def. 3) For all real-valued finite sequences x, y such that x, $y \in B_0$ and $x \neq y$ holds $|(x, y)| = 0$.

Let us observe that every set which is empty is also $\mathbb{R}$-orthogonal.

We now state the proposition

(16) B_0 is $\mathbb{R}$-orthogonal if and only if for all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ such that x, $y \in B_0$ and $x \neq y$ holds x, y are orthogonal.

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-normal if and only if:

(Def. 4) For every real-valued finite sequence x such that $x \in B_0$ holds $|x| = 1$.

Let us observe that every set which is empty is also $\mathbb{R}$-normal.

Let us observe that there exists a set which is $\mathbb{R}$-normal.

Let B_0, B_1 be $\mathbb{R}$-normal sets. One can verify that $B_0 \cup B_1$ is $\mathbb{R}$-normal.

One can prove the following propositions:

(17) If $|f| = 1$, then $\{f\}$ is $\mathbb{R}$-normal.

(18) If B_0 is $\mathbb{R}$-normal and $|x_0| = 1$, then $B_0 \cup \{x_0\}$ is $\mathbb{R}$-normal.

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-orthonormal if and only if:

(Def. 5) B_0 is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal.

Let us note that every set which is $\mathbb{R}$-orthonormal is also $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal and every set which is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal is also $\mathbb{R}$-orthonormal.

Let us observe that $\{\langle 1 \rangle\}$ is $\mathbb{R}$-orthonormal.

Let us observe that there exists a set which is $\mathbb{R}$-orthonormal and non empty.

Let us consider n. One can verify that there exists a subset of $\mathcal{R}^n$ which is $\mathbb{R}$-orthonormal.

Let us consider n and let B_0 be a subset of $\mathcal{R}^n$. We say that B_0 is complete if and only if:

(Def. 6) For every $\mathbb{R}$-orthonormal subset B of $\mathcal{R}^n$ such that $B_0 \subseteq B$ holds $B = B_0$.

Let n be an element of $\mathbb{N}$ and let B_0 be a subset of $\mathcal{R}^n$. We say that B_0 is orthogonal basis if and only if:

(Def. 7) B_0 is $\mathbb{R}$-orthonormal and complete.

Let us consider n. One can verify that every subset of $\mathcal{R}^n$ which is orthogonal basis is also $\mathbb{R}$-orthonormal and complete and every subset of $\mathcal{R}^n$ which is $\mathbb{R}$-orthonormal and complete is also orthogonal basis.

The following propositions are true:

(19) For every subset B_0 of $\mathcal{R}^0$ such that B_0 is orthogonal basis holds $B_0 = \emptyset$.

(20) Let B_0 be a subset of $\mathcal{R}^n$ and y be an element of $\mathcal{R}^n$. Suppose B_0 is orthogonal basis and for every element x of $\mathcal{R}^n$ such that $x \in B_0$ holds $|(x, y)| = 0$. Then $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

3. Linear Manifolds

Let us consider n and let X be a subset of $\mathcal{R}^n$. We say that X is linear manifold if and only if:

(Def. 8) For all elements x, y of $\mathcal{R}^n$ and for all elements a, b of $\mathbb{R}$ such that x, $y \in X$ holds $a \cdot x + b \cdot y \in X$.

Let us consider n. Observe that $\Omega_{\mathcal{R}^n}$ is linear manifold.

The following proposition is true

(21) $\{\langle \underbrace{0, \ldots, 0}_{n} \rangle\}$ is linear manifold.

Let us consider n. Observe that $\{\langle \underbrace{0, \ldots, 0}_{n} \rangle\}$ is linear manifold.

Let us consider n and let X be a subset of $\mathcal{R}^n$. The linear span of X yielding a subset of $\mathcal{R}^n$ is defined by:

(Def. 9) The linear span of $X = \bigcap\{Y \subseteq \mathcal{R}^n$: Y is linear manifold $\wedge$ $X \subseteq Y\}$.

Let us consider n and let X be a subset of $\mathcal{R}^n$. Observe that the linear span of X is linear manifold.

Let us consider n and let f be a finite sequence of elements of $\mathcal{R}^n$. The functor $\sum f$ yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 10)(i) There exists a finite sequence g of elements of $\mathcal{R}^n$ such that $\operatorname{len} f = \operatorname{len} g$ and $f(1) = g(1)$ and for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $g(i+1) = g_i + f_{i+1}$ and $\sum f = g(\operatorname{len} f)$ if $\operatorname{len} f > 0$,

(ii) $\sum f = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, otherwise.

Let n be a natural number and let f be a finite sequence of elements of $\mathcal{R}^n$. The functor $\operatorname{accum} f$ yields a finite sequence of elements of $\mathcal{R}^n$ and is defined as follows:

(Def. 11) $\operatorname{len} f = \operatorname{len} \operatorname{accum} f$ and $f(1) = (\operatorname{accum} f)(1)$ and for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $(\operatorname{accum} f)(i+1) = (\operatorname{accum} f)_i + f_{i+1}$.

We now state several propositions:

(22) For every finite sequence f of elements of $\mathcal{R}^n$ such that $\operatorname{len} f > 0$ holds $(\operatorname{accum} f)(\operatorname{len} f) = \sum f$.

(23) For all finite sequences F, F_2 of elements of $\mathcal{R}^n$ and for every permutation h of $\operatorname{dom} F$ such that $F_2 = F \circ h$ holds $\sum F_2 = \sum F$.

(24) For every element k of $\mathbb{N}$ holds $\sum k \mapsto \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(25) Let g be a finite sequence of elements of $\mathcal{R}^n$, h be a finite sequence of elements of $\mathbb{N}$, and F be a finite sequence of elements of $\mathcal{R}^n$. Suppose h is increasing and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $F = g \cdot h$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} g$ and $i \notin \operatorname{rng} h$ holds $g(i) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then $\sum g = \sum F$.

(26) Let g be a finite sequence of elements of $\mathcal{R}^n$, h be a finite sequence of elements of $\mathbb{N}$, and F be a finite sequence of elements of $\mathcal{R}^n$. Suppose h is one-to-one and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $F = g \cdot h$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} g$ and $i \notin \operatorname{rng} h$ holds $g(i) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then $\sum g = \sum F$.

4. Standard Basis

Let us consider n, i. Then the base finite sequence of n and i is an element of $\mathcal{R}^n$.

The following propositions are true:

(27) Let i_1, i_2 be elements of $\mathbb{N}$. Suppose that

(i) $1 \leq i_1$,

(ii) $i_1 \leq n$,

(iii) $1 \leq i_2$,

(iv) $i_2 \leq n$, and

(v) the base finite sequence of n and i_1 = the base finite sequence of n and i_2.

Then $i_1 = i_2$.

(28) ^{2}(the base finite sequence of n and i) = the base finite sequence of n and i.

(29) If $1 \leq i \leq n$, then $\sum$ the base finite sequence of n and $i = 1$.

(30) If $1 \leq i \leq n$, then |the base finite sequence of n and $i| = 1$.

(31) Suppose $1 \leq i \leq n$ and $1 \leq j \leq n$ and $i \neq j$. Then |(the base finite sequence of n and i, the base finite sequence of n and j)| = 0.

(32) For every element x of $\mathcal{R}^n$ such that $1 \leq i \leq n$ holds $|(x,$ the base finite sequence of n and $i)| = x(i)$.

Let us consider n and let x_0 be an element of $\mathcal{R}^n$. The functor $\operatorname{ProjFinSeq} x_0$ yields a finite sequence of elements of $\mathcal{R}^n$ and is defined by the conditions (Def. 12).

(Def. 12)(i) $\operatorname{len} \operatorname{ProjFinSeq} x_0 = n$, and

(ii) for every i such that $1 \leq i \leq n$ holds $(\operatorname{ProjFinSeq} x_0)(i) = |(x_0,$ the base finite sequence of n and $i)| \cdot$ the base finite sequence of n and i.

The following proposition is true

(33) For every element x_0 of $\mathcal{R}^n$ holds $x_0 = \sum \mathrm{ProjFinSeq}\, x_0$.

Let us consider n. The functor $\mathbb{R}$N-Base n yields a subset of $\mathcal{R}^n$ and is defined by:

(Def. 13) $\mathbb{R}$N-Base n = {the base finite sequence of n and i; i ranges over elements of $\mathbb{N}$: $1 \leq i \ \wedge \ i \leq n$}.

Next we state the proposition

(34) For every non zero element n of $\mathbb{N}$ holds $\mathbb{R}$N-Base $n \neq \emptyset$.

Let us mention that $\mathbb{R}$N-Base 0 is empty.

Let n be a non zero element of $\mathbb{N}$. Note that $\mathbb{R}$N-Base n is non empty.

Let us consider n. Observe that $\mathbb{R}$N-Base n is orthogonal basis.

Let us consider n. Observe that there exists a subset of $\mathcal{R}^n$ which is orthogonal basis.

Let us consider n. An orthogonal basis of n is an orthogonal basis subset of $\mathcal{R}^n$.

Let n be a non zero element of $\mathbb{N}$. Observe that every orthogonal basis of n is non empty.

5. Finite Real Unitary Spaces and Finite Real Linear Spaces

Let n be an element of $\mathbb{N}$. Observe that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is constituted finite sequences. Let n be an element of $\mathbb{N}$. One can check that every element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is real-valued.

Let n be an element of $\mathbb{N}$, let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can verify that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let n be an element of $\mathbb{N}$, let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, let y be a real-valued function, and let a, b be elements of $\mathbb{R}$. Observe that $a \cdot x$ and $b\, y$ can be identified when $a = b$ and $x = y$.

Let n be an element of $\mathbb{N}$, let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a be a real-valued function. Observe that $-x$ and $-a$ can be identified when $x = a$.

Let n be an element of $\mathbb{N}$, let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can check that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$. The following three propositions are true:

(35) Let n be an element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and u, v be points of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $x = u$ and $y = v$, then $\otimes_{\mathcal{E}^n}(\langle\!\langle u, v \rangle\!\rangle) = |(x, y)|$.

(36) Let n, j be elements of $\mathbb{N}$, F be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, B_2 be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, v_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and l be a linear combination of B_2. Suppose F is one-to-one and B_2 is $\mathbb{R}$-orthogonal and $\mathrm{rng}\, F$ = the support of l and $v_0 \in B_2$ and $j \in \mathrm{dom}(l\, F)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle\!\langle v_0, \sum l\, F \rangle\!\rangle) = \otimes_{\mathcal{E}^n}(\langle\!\langle v_0, l(F_j) \cdot v_0 \rangle\!\rangle)$.

(37) Let n be an element of $\mathbb{N}$, f be a finite sequence of elements of $\mathcal{R}^n$, and g be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $f = g$, then $\sum f = \sum g$.

Let A be a set. Note that $\mathbb{R}^A_{\mathbb{R}}$ is constituted functions.

Let us consider n. Observe that $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ is constituted finite sequences.

Let A be a set. One can verify that every element of $\mathbb{R}^A_{\mathbb{R}}$ is real-valued.

Let A be a set, let x, y be vectors of $\mathbb{R}^A_{\mathbb{R}}$, and let a, b be real-valued functions. Observe that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let A be a set, let x be a vector of $\mathbb{R}^A_{\mathbb{R}}$, let y be a real-valued function, and let a, b be elements of $\mathbb{R}$. Observe that $a \cdot x$ and $b\, y$ can be identified when $a = b$ and $x = y$.

Let A be a set, let x be a vector of $\mathbb{R}^A_{\mathbb{R}}$, and let a be a real-valued function. One can check that $-x$ and $-a$ can be identified when $x = a$.

Let A be a set, let x, y be vectors of $\mathbb{R}^A_{\mathbb{R}}$, and let a, b be real-valued functions. Observe that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

The following propositions are true:

(38) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x be an element of $\mathcal{R}^n$, and a be a real number. If $x \in$ the carrier of X, then $a \cdot x \in$ the carrier of X.

(39) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ and x, y be elements of $\mathcal{R}^n$. Suppose $x \in$ the carrier of X and $y \in$ the carrier of X. Then $x + y \in$ the carrier of X.

(40) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x, y be elements of $\mathcal{R}^n$, and a, b be real numbers. Suppose $x \in$ the carrier of X and $y \in$ the carrier of X. Then $a \cdot x + b \cdot y \in$ the carrier of X.

(41) For all elements x, y of $\mathcal{R}^n$ and for all points u, v of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ such that $x = u$ and $y = v$ holds $\otimes_{\mathcal{E}^n}(\langle u, v \rangle) = |(x, y)|$.

(42) Let F be a finite sequence of elements of the carrier of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, B_2 be a subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, v_0 be an element of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, and l be a linear combination of B_2. Suppose F is one-to-one and B_2 is $\mathbb{R}$-orthogonal and $\mathrm{rng}\, F =$ the support of l and $v_0 \in B_2$ and $j \in \mathrm{dom}(l\, F)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle v_0, \sum l\, F \rangle) = \otimes_{\mathcal{E}^n}(\langle v_0, l(F_j) \cdot v_0 \rangle)$.

Let us consider n. Note that every subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ which is $\mathbb{R}$-orthonormal is also linearly independent.

Let n be an element of $\mathbb{N}$. Note that every subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ which is $\mathbb{R}$-orthonormal is also linearly independent. Next we state the proposition

(43) Let B_2 be a subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x, y be elements of $\mathcal{R}^n$, and a be a real number. If B_2 is linearly independent and $x, y \in B_2$ and $y = a \cdot x$, then $x = y$.

6. Finite Dimensionality of the Spaces

Let us consider n. One can check that $\mathbb{R}$N-Base n is finite.

The following propositions are true:

(44) card $\mathbb{R}$N-Base $n = n$.

(45) Let f be a finite sequence of elements of $\mathcal{R}^n$ and g be a finite sequence of elements of the carrier of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$. If $f = g$, then $\sum f = \sum g$.

(46) Let x_0 be an element of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ and B be a subset of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$. If $B =$ $\mathbb{R}$N-Base n, then there exists a linear combination l of B such that $x_0 = \sum l$.

(47) Let n be an element of $\mathbb{N}$, x_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and B be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $B = \mathbb{R}$N-Base n, then there exists a linear combination l of B such that $x_0 = \sum l$.

(48) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ such that $B = \mathbb{R}$N-Base n holds B is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$.

Let us consider n. Observe that $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ is finite dimensional.

We now state several propositions:

(49) $\dim(\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}) = n$.

(50) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ such that B is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ holds $\overline{\overline{B}} = n$.

(51) $\emptyset$ is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,0}$.

(52) For every element n of $\mathbb{N}$ holds $\mathbb{R}$N-Base n is a basis of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$.

(53) Every orthogonal basis of n is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$.

Let n be an element of $\mathbb{N}$. Note that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is finite dimensional.

We now state two propositions:

(54) For every element n of $\mathbb{N}$ holds $\dim(\langle \mathcal{E}^n, (\cdot|\cdot) \rangle) = n$.

(55) For every orthogonal basis B of n holds $\overline{\overline{B}} = n$.

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Received September 23, 2008