

Several Integrability Formulas of Some Functions, Orthogonal Polynomials and Norm Functions

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Summary. In this article, we give several integrability formulas of some functions including the trigonometric function and the index function [3]. We also give the definitions of the orthogonal polynomial and norm function, and some of their important properties [19].

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The terminology and notation used here are introduced in the following articles: [10], [21], [17], [6], [20], [1], [9], [13], [2], [4], [18], [15], [5], [8], [11], [14], [12], [16], and [7].

For simplicity, we use the following convention: r, p, x denote real numbers, n denotes an element of \mathbb{N} , A denotes a closed-interval subset of \mathbb{R} , f, g denote partial functions from \mathbb{R} to \mathbb{R} , and Z denotes an open subset of \mathbb{R} .

We now state a number of propositions:

- (1) $-(\text{the function exp}) \cdot ((-1)\square+0)$ is differentiable on \mathbb{R} and for every x holds $(-(\text{the function exp}) \cdot ((-1)\square+0))'_{\mathbb{R}}(x) = \exp(-x)$.

- (2) $\int_A ((\text{the function exp}) \cdot ((-1)\square+0))(x)dx = -\exp(-\sup A) + \exp(-\inf A).$
- (3) $\frac{1}{2} ((\text{the function exp}) \cdot (2\square+0))$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{2} ((\text{the function exp}) \cdot (2\square+0)))'_{\mathbb{R}}(x) = \exp(2 \cdot x).$
- (4) $\int_A ((\text{the function exp}) \cdot (2\square+0))(x)dx = \frac{1}{2} \cdot \exp(2 \cdot \sup A) - \frac{1}{2} \cdot \exp(2 \cdot \inf A).$
- (5) Suppose $r \neq 0$. Then $\frac{1}{r} ((\text{the function exp}) \cdot (r\square+0))$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{r} ((\text{the function exp}) \cdot (r\square+0)))'_{\mathbb{R}}(x) = \exp(r \cdot x).$
- (6) If $r \neq 0$, then $\int_A ((\text{the function exp}) \cdot (r\square+0))(x)dx = \frac{1}{r} \cdot \exp(r \cdot \sup A) - \frac{1}{r} \cdot \exp(r \cdot \inf A).$
- (7) $\int_A ((\text{the function sin}) \cdot (2\square+0))(x)dx = (-\frac{1}{2}) \cdot \cos(2 \cdot \sup A) - (-\frac{1}{2}) \cdot \cos(2 \cdot \inf A).$
- (8) Suppose $n \neq 0$. Then $(-\frac{1}{n}) ((\text{the function cos}) \cdot (n\square+0))$ is differentiable on \mathbb{R} and for every x holds $((-\frac{1}{n}) ((\text{the function cos}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = \sin(n \cdot x).$
- (9) If $n \neq 0$, then $\int_A ((\text{the function sin}) \cdot (n\square+0))(x)dx = (-\frac{1}{n}) \cdot \cos(n \cdot \sup A) - (-\frac{1}{n}) \cdot \cos(n \cdot \inf A).$
- (10) $\frac{1}{2} ((\text{the function sin}) \cdot (2\square+0))$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{2} ((\text{the function sin}) \cdot (2\square+0)))'_{\mathbb{R}}(x) = \cos(2 \cdot x).$
- (11) $\int_A ((\text{the function cos}) \cdot (2\square+0))(x)dx = \frac{1}{2} \cdot \sin(2 \cdot \sup A) - \frac{1}{2} \cdot \sin(2 \cdot \inf A).$
- (12) Suppose $n \neq 0$. Then $\frac{1}{n} ((\text{the function sin}) \cdot (n\square+0))$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{n} ((\text{the function sin}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = \cos(n \cdot x).$
- (13) If $n \neq 0$, then $\int_A ((\text{the function cos}) \cdot (n\square+0))(x)dx = \frac{1}{n} \cdot \sin(n \cdot \sup A) - \frac{1}{n} \cdot \sin(n \cdot \inf A).$
- (14) If $A \subseteq Z$, then $\int_A (\text{id}_Z (\text{the function sin}))(x)dx = ((-\sup A) \cdot \cos \sup A + \sin \sup A) - ((-\inf A) \cdot \cos \inf A + \sin \inf A).$
- (15) If $A \subseteq Z$, then $\int_A (\text{id}_Z (\text{the function cos}))(x)dx = (\sup A \cdot \sin \sup A + \cos \sup A) - (\inf A \cdot \sin \inf A + \cos \inf A).$

- (16) id_Z (the function \cos) is differentiable on Z and for every x such that $x \in Z$ holds $(\text{id}_Z (\text{the function } \cos))'_{\uparrow Z}(x) = \cos x - x \cdot \sin x$.
- (17)(i) $-\text{the function } \sin + \text{id}_Z$ (the function \cos) is differentiable on Z , and
(ii) for every x such that $x \in Z$ holds $(-\text{the function } \sin + \text{id}_Z (\text{the function } \cos))'_{\uparrow Z}(x) = -x \cdot \sin x$.
- (18) If $A \subseteq Z$, then $\int_A ((-\text{id}_Z) (\text{the function } \sin))(x)dx = (-\sin \sup A + \sup A \cdot \cos \sup A) - (-\sin \inf A + \inf A \cdot \cos \inf A)$.
- (19)(i) $-\text{the function } \cos - \text{id}_Z$ (the function \sin) is differentiable on Z , and
(ii) for every x such that $x \in Z$ holds $(-\text{the function } \cos - \text{id}_Z (\text{the function } \sin))'_{\uparrow Z}(x) = -x \cdot \cos x$.
- (20) If $A \subseteq Z$, then $\int_A ((-\text{id}_Z) (\text{the function } \cos))(x)dx = -\cos \sup A - \sup A \cdot \sin \sup A - (-\cos \inf A - \inf A \cdot \sin \inf A)$.
- (21) If $A \subseteq Z$, then $\int_A ((\text{the function } \sin) + \text{id}_Z (\text{the function } \cos))(x)dx = \sup A \cdot \sin \sup A - \inf A \cdot \sin \inf A$.
- (22) If $A \subseteq Z$, then $\int_A (-\text{the function } \cos + \text{id}_Z (\text{the function } \sin))(x)dx = (-\sup A) \cdot \cos \sup A - (-\inf A) \cdot \cos \inf A$.
- (23) $\int_A ((1 \square + 0) (\text{the function } \exp))(x)dx = \exp(\sup A - 1) - \exp(\inf A - 1)$.
- (24) $\frac{1}{n+1} (\square^{n+1})$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{n+1} (\square^{n+1}))'_{\uparrow \mathbb{R}}(x) = x^n$.
- (25) $\int_A (\square^n)(x)dx = \frac{1}{n+1} \cdot (\sup A)^{n+1} - \frac{1}{n+1} \cdot (\inf A)^{n+1}$.
- (26) For all partial functions f, g from \mathbb{R} to \mathbb{R} and for every non empty subset C of \mathbb{R} holds $(f - g) \uparrow C = f \uparrow C - g \uparrow C$.
- (27) For all partial functions f_1, f_2, g from \mathbb{R} to \mathbb{R} and for every non empty subset C of \mathbb{R} holds $((f_1 + f_2) \uparrow C) (g \uparrow C) = (f_1 g + f_2 g) \uparrow C$.
- (28) For all partial functions f_1, f_2, g from \mathbb{R} to \mathbb{R} and for every non empty subset C of \mathbb{R} holds $((f_1 - f_2) \uparrow C) (g \uparrow C) = (f_1 g - f_2 g) \uparrow C$.
- (29) For all partial functions f_1, f_2, g from \mathbb{R} to \mathbb{R} and for every non empty subset C of \mathbb{R} holds $((f_1 f_2) \uparrow C) (g \uparrow C) = (f_1 \uparrow C) ((f_2 g) \uparrow C)$.

Let A be a closed-interval subset of \mathbb{R} and let f, g be partial functions from \mathbb{R} to \mathbb{R} . The functor $\langle f, g \rangle_A$ yielding a real number is defined by:

$$\text{(Def. 1)} \quad \langle f, g \rangle_A = \int_A (f g)(x)dx.$$

The following propositions are true:

- (30) For all partial functions f, g from \mathbb{R} to \mathbb{R} and for every closed-interval subset A of \mathbb{R} holds $\langle f, g \rangle_A = \langle g, f \rangle_A$.
- (31) Let f_1, f_2, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose that
- (i) $(f_1 g) \upharpoonright A$ is total,
 - (ii) $(f_2 g) \upharpoonright A$ is total,
 - (iii) $(f_1 g) \upharpoonright A$ is bounded,
 - (iv) $f_1 g$ is integrable on A ,
 - (v) $(f_2 g) \upharpoonright A$ is bounded, and
 - (vi) $f_2 g$ is integrable on A .
- Then $\langle f_1 + f_2, g \rangle_A = \langle (f_1), g \rangle_A + \langle (f_2), g \rangle_A$.
- (32) Let f_1, f_2, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose that
- (i) $(f_1 g) \upharpoonright A$ is total,
 - (ii) $(f_2 g) \upharpoonright A$ is total,
 - (iii) $(f_1 g) \upharpoonright A$ is bounded,
 - (iv) $f_1 g$ is integrable on A ,
 - (v) $(f_2 g) \upharpoonright A$ is bounded, and
 - (vi) $f_2 g$ is integrable on A .
- Then $\langle f_1 - f_2, g \rangle_A = \langle (f_1), g \rangle_A - \langle (f_2), g \rangle_A$.
- (33) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose $(f g) \upharpoonright A$ is bounded and $f g$ is integrable on A and $A \subseteq \text{dom}(f g)$. Then $\langle -f, g \rangle_A = -\langle f, g \rangle_A$.
- (34) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose $(f g) \upharpoonright A$ is bounded and $f g$ is integrable on A and $A \subseteq \text{dom}(f g)$. Then $\langle r f, g \rangle_A = r \cdot \langle f, g \rangle_A$.
- (35) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose $(f g) \upharpoonright A$ is bounded and $f g$ is integrable on A and $A \subseteq \text{dom}(f g)$. Then $\langle r f, p g \rangle_A = r \cdot p \cdot \langle f, g \rangle_A$.
- (36) For all partial functions f, g, h from \mathbb{R} to \mathbb{R} and for every closed-interval subset A of \mathbb{R} holds $\langle f g, h \rangle_A = \langle f, g h \rangle_A$.
- (37) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose that $(f f) \upharpoonright A$ is total and $(f g) \upharpoonright A$ is total and $(g g) \upharpoonright A$ is total and $(f f) \upharpoonright A$ is bounded and $(f g) \upharpoonright A$ is bounded and $(g g) \upharpoonright A$ is bounded and $f f$ is integrable on A and $f g$ is integrable on A and $g g$ is integrable on A . Then $\langle f + g, f + g \rangle_A = \langle f, f \rangle_A + 2 \cdot \langle f, g \rangle_A + \langle g, g \rangle_A$.

Let A be a closed-interval subset of \mathbb{R} and let f, g be partial functions from \mathbb{R} to \mathbb{R} . We say that f is orthogonal with g in A if and only if:

(Def. 2) $\langle f, g \rangle_A = 0$.

The following propositions are true:

- (38) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose that $(f f) \upharpoonright A$ is total and $(f g) \upharpoonright A$ is total and $(g g) \upharpoonright A$ is total and $(f f) \upharpoonright A$ is bounded and $(f g) \upharpoonright A$ is bounded and $(g g) \upharpoonright A$ is bounded and $f f$ is integrable on A and $f g$ is integrable on A and $g g$ is integrable on A and f is orthogonal with g in A . Then $\langle f + g, f + g \rangle_A = \langle f, f \rangle_A + \langle g, g \rangle_A$.
- (39) Let f be a partial function from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose $(f f) \upharpoonright A$ is total and $(f f) \upharpoonright A$ is bounded and $f f$ is integrable on A and for every x such that $x \in A$ holds $((f f) \upharpoonright A)(x) \geq 0$. Then $\langle f, f \rangle_A \geq 0$.
- (40) The function \sin is orthogonal with the function \cos in $[0, \pi]$.
- (41) The function \sin is orthogonal with the function \cos in $[0, \pi \cdot 2]$.
- (42) The function \sin is orthogonal with the function \cos in $[2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$.
- (43) The function \sin is orthogonal with the function \cos in $[x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$.
- (44) The function \sin is orthogonal with the function \cos in $[-\pi, \pi]$.
- (45) The function \sin is orthogonal with the function \cos in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- (46) The function \sin is orthogonal with the function \cos in $[-2 \cdot \pi, 2 \cdot \pi]$.
- (47) The function \sin is orthogonal with the function \cos in $[-2 \cdot n \cdot \pi, 2 \cdot n \cdot \pi]$.
- (48) The function \sin is orthogonal with the function \cos in $[x - 2 \cdot n \cdot \pi, x + 2 \cdot n \cdot \pi]$.

Let A be a closed-interval subset of \mathbb{R} and let f be a partial function from \mathbb{R} to \mathbb{R} . The functor $\|f\|_A$ yields a real number and is defined by:

(Def. 3) $\|f\|_A = \sqrt{\langle f, f \rangle_A}$.

Next we state three propositions:

- (49) Let f be a partial function from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose $(f f) \upharpoonright A$ is total and $(f f) \upharpoonright A$ is bounded and $f f$ is integrable on A and for every x such that $x \in A$ holds $((f f) \upharpoonright A)(x) \geq 0$. Then $0 \leq \|f\|_A$.
- (50) For every partial function f from \mathbb{R} to \mathbb{R} and for every closed-interval subset A of \mathbb{R} holds $\|1 f\|_A = \|f\|_A$.
- (51) Let f, g be partial functions from \mathbb{R} to \mathbb{R} and A be a closed-interval subset of \mathbb{R} . Suppose that $(f f) \upharpoonright A$ is total and $(f g) \upharpoonright A$ is total and $(g g) \upharpoonright A$ is total and $(f f) \upharpoonright A$ is bounded and $(f g) \upharpoonright A$ is bounded and $(g g) \upharpoonright A$ is bounded and $f f$ is integrable on A and $f g$ is integrable on A and $g g$ is integrable on A and f is orthogonal with g in A and for every x such that $x \in A$ holds $((f f) \upharpoonright A)(x) \geq 0$ and for every x such that $x \in A$ holds $((g g) \upharpoonright A)(x) \geq 0$. Then $(\|f + g\|_A)^2 = (\|f\|_A)^2 + (\|g\|_A)^2$.

For simplicity, we follow the rules: a, b, x are real numbers, n is an element of \mathbb{N} , A is a closed-interval subset of \mathbb{R} , f, f_1, f_2 are partial functions from \mathbb{R} to \mathbb{R} , and Z is an open subset of \mathbb{R} .

Next we state several propositions:

(52) If $-a \notin A$, then $\frac{1}{1 \square + a} \upharpoonright A$ is continuous.

(53) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $Z = \text{dom } f$,

(iv) $\text{dom } f = \text{dom } f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(a+x)^2}$, and

(vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = f(\sup A)^{-1} - f(\inf A)^{-1}.$$

(54) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $\text{dom}((-1) \frac{1}{f}) = Z$,

(iv) $\text{dom}((-1) \frac{1}{f}) = \text{dom } f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a+x)^2}$, and

(vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = -f(\sup A)^{-1} + f(\inf A)^{-1}.$$

(55) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) \neq 0$,

(iii) $\text{dom } f = Z$,

(iv) $\text{dom } f = \text{dom } f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a-x)^2}$, and

(vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = f(\sup A)^{-1} - f(\inf A)^{-1}.$$

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,

(iii) $\text{dom}((\text{the function } \ln) \cdot f) = Z$,

(iv) $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a+x}$, and

(vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln(a + \sup A) - \ln(a + \inf A).$$

Next we state a number of propositions:

(57) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = x - a$ and $f(x) > 0$,
- (iii) $\text{dom}((\text{the function } \ln) \cdot f) = Z$,
- (iv) $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x-a}$, and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln f(\sup A) - \ln f(\inf A).$$

(58) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
- (iii) $\text{dom}(-(\text{the function } \ln) \cdot f) = Z$,
- (iv) $\text{dom}(-(\text{the function } \ln) \cdot f) = \text{dom } f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a-x}$, and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = -\ln(a - \sup A) + \ln(a - \inf A).$$

(59) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z - a f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x}{a+x}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x) dx = \sup A - a \cdot f(\sup A) - (\inf A - a \cdot f(\inf A))$.

(60) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\text{dom}((2 \cdot a) f - \text{id}_Z) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{a-x}{a+x}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x) dx = 2 \cdot a \cdot f(\sup A) - \sup A - (2 \cdot a \cdot f(\inf A) - \inf A)$.

(61) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z - (2 \cdot a) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+a}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x) dx = \sup A - 2 \cdot a \cdot f(\sup A) - (\inf A - 2 \cdot a \cdot f(\inf A))$.

(62) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z + (2 \cdot a) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-a}$ and $f_2 \upharpoonright A$

is continuous. Then $\int_A f_2(x)dx = (\sup A + 2 \cdot a \cdot f(\sup A)) - (\inf A + 2 \cdot a \cdot f(\inf A))$.

(63) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z + (a - b) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x+b}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (a - b) \cdot f(\sup A)) - (\inf A + (a - b) \cdot f(\inf A))$.

(64) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z + (a + b) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-b}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (a + b) \cdot f(\sup A)) - (\inf A + (a + b) \cdot f(\inf A))$.

(65) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z - (a + b) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+b}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x)dx = \sup A - (a + b) \cdot f(\sup A) - (\inf A - (a + b) \cdot f(\inf A))$.

(66) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\text{dom}(\text{id}_Z + (b - a) f) = Z = \text{dom } f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x-b}$ and $f_2 \upharpoonright A$ is continuous. Then $\int_A f_2(x)dx = (\sup A + (b - a) \cdot f(\sup A)) - (\inf A + (b - a) \cdot f(\inf A))$.

(67) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$,
- (iii) $\text{dom}((\text{the function } \ln) \cdot f) = Z$,
- (iv) $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x}$, and
- (vi) $f_2 \upharpoonright A$ is continuous.

Then $\int_A f_2(x)dx = \ln \sup A - \ln \inf A$.

(68) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $x > 0$,
- (iii) $\text{dom}((\text{the function } \ln) \cdot (\square^n)) = Z$,

- (iv) $\text{dom}((\text{the function } \ln) \cdot (\square^n)) = \text{dom } f_2,$
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{n}{x},$ and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln((\sup A)^n) - \ln((\inf A)^n).$$

(69) Suppose that

- (i) $A \subseteq Z,$
- (ii) for every x such that $x \in Z$ holds $f(x) = x,$
- (iii) $\text{dom}((\text{the function } \ln) \cdot \frac{1}{f}) = Z,$
- (iv) $\text{dom}((\text{the function } \ln) \cdot \frac{1}{f}) = \text{dom } f_2,$
- (v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{x},$ and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = -\ln \sup A + \ln \inf A.$$

(70) Suppose that

- (i) $A \subseteq Z,$
- (ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0,$
- (iii) $\text{dom}(\frac{2}{3} f^{\frac{3}{2}}) = Z,$
- (iv) $\text{dom}(\frac{2}{3} f^{\frac{3}{2}}) = \text{dom } f_2,$
- (v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{\frac{1}{2}},$ and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = \frac{2}{3} \cdot (a + \sup A)^{\frac{3}{2}} - \frac{2}{3} \cdot (a + \inf A)^{\frac{3}{2}}.$$

(71) Suppose that

- (i) $A \subseteq Z,$
- (ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0,$
- (iii) $\text{dom}((-\frac{2}{3}) f^{\frac{3}{2}}) = Z,$
- (iv) $\text{dom}((-\frac{2}{3}) f^{\frac{3}{2}}) = \text{dom } f_2,$
- (v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{\frac{1}{2}},$ and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x) dx = -\frac{2}{3} \cdot (a - \sup A)^{\frac{3}{2}} + \frac{2}{3} \cdot (a - \inf A)^{\frac{3}{2}}.$$

(72) Suppose that

- (i) $A \subseteq Z,$
- (ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0,$
- (iii) $\text{dom}(2 f^{\frac{1}{2}}) = Z,$
- (iv) $\text{dom}(2 f^{\frac{1}{2}}) = \text{dom } f_2,$
- (v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{-\frac{1}{2}},$ and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x)dx = 2 \cdot (a + \sup A)^{\frac{1}{2}} - 2 \cdot (a + \inf A)^{\frac{1}{2}}.$$

(73) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
- (iii) $\text{dom}((-2) f^{\frac{1}{2}}) = Z$,
- (iv) $\text{dom}((-2) f^{\frac{1}{2}}) = \text{dom } f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{-\frac{1}{2}}$, and
- (vi) $f_2 \upharpoonright A$ is continuous.

$$\text{Then } \int_A f_2(x)dx = -2 \cdot (a - \sup A)^{\frac{1}{2}} + 2 \cdot (a - \inf A)^{\frac{1}{2}}.$$

(74) Suppose that

- (i) $A \subseteq Z$,
- (ii) $\text{dom}((-id_Z)(\text{the function cos}) + \text{the function sin}) = Z$,
- (iii) for every x such that $x \in Z$ holds $f(x) = x \cdot \sin x$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x)dx = (-\sup A \cdot \cos \sup A + \sin \sup A) - (-\inf A \cdot \cos \inf A + \sin \inf A).$$

(75) Suppose $A \subseteq Z$ and $\text{dom}(\text{the function sec}) = Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x}{(\cos x)^2}$ and $Z = \text{dom } f$ and $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x)dx = \sec \sup A - \sec \inf A.$$

(76) Suppose $Z \subseteq \text{dom}(-\text{the function cosec})$. Then $-\text{the function cosec}$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\text{the function cosec})' \upharpoonright_Z(x) = \frac{\cos x}{(\sin x)^2}$.

(77) Suppose $A \subseteq Z$ and $\text{dom}(-\text{the function cosec}) = Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\cos x}{(\sin x)^2}$ and $Z = \text{dom } f$ and $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x)dx = -\text{cosec } \sup A + \text{cosec } \inf A.$$

REFERENCES

- [1] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [2] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in \mathcal{E}^2 . *Formalized Mathematics*, 6(3):427–440, 1997.
- [3] Chuazhang Chen. *Mathematical Analysis*. Higher Education Press, Beijing, 1978.
- [4] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(1):93–102, 1999.
- [5] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from \mathbb{R} to \mathbb{R} and integrability for continuous functions. *Formalized Mathematics*, 9(2):281–284, 2001.

- [6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [8] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(4):697–702, 1990.
- [9] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(4):703–709, 1990.
- [10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [11] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(1):125–130, 1991.
- [12] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.
- [13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [14] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(2):195–200, 2004.
- [15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [16] Andrzej Trybulec and Yatsuka Nakamura. On the decomposition of a simple closed curve into two arcs. *Formalized Mathematics*, 10(3):163–167, 2002.
- [17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [18] Peng Wang and Bo Li. Several differentiation formulas of special functions. Part V. *Formalized Mathematics*, 15(3):73–79, 2007, doi:10.2478/v10037-007-0009-4.
- [19] Renhong Wang. *Numerical approximation*. Higher Education Press, Beijing, 1999.
- [20] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [21] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(2):255–263, 1998.

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