

# Several Integrability Formulas of Some Functions, Orthogonal Polynomials and Norm Functions

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**Summary.** In this article, we give several integrability formulas of some functions including the trigonometric function and the index function [3]. We also give the definitions of the orthogonal polynomial and norm function, and some of their important properties [19].

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The terminology and notation used here are introduced in the following articles: [10], [21], [17], [6], [20], [1], [9], [13], [2], [4], [18], [15], [5], [8], [11], [14], [12], [16], and [7].

For simplicity, we use the following convention:  $r, p, x$  denote real numbers,  $n$  denotes an element of  $\mathbb{N}$ ,  $A$  denotes a closed-interval subset of  $\mathbb{R}$ ,  $f, g$  denote partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ , and  $Z$  denotes an open subset of  $\mathbb{R}$ .

We now state a number of propositions:

- (1)  $-(\text{the function exp}) \cdot ((-1)\square+0)$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(-(\text{the function exp}) \cdot ((-1)\square+0))'_{\mathbb{R}}(x) = \exp(-x)$ .

- (2)  $\int_A ((\text{the function exp}) \cdot ((-1)\square+0))(x)dx = -\exp(-\sup A) + \exp(-\inf A).$
- (3)  $\frac{1}{2} ((\text{the function exp}) \cdot (2\square+0))$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(\frac{1}{2} ((\text{the function exp}) \cdot (2\square+0)))'_{\mathbb{R}}(x) = \exp(2 \cdot x).$
- (4)  $\int_A ((\text{the function exp}) \cdot (2\square+0))(x)dx = \frac{1}{2} \cdot \exp(2 \cdot \sup A) - \frac{1}{2} \cdot \exp(2 \cdot \inf A).$
- (5) Suppose  $r \neq 0$ . Then  $\frac{1}{r} ((\text{the function exp}) \cdot (r\square+0))$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(\frac{1}{r} ((\text{the function exp}) \cdot (r\square+0)))'_{\mathbb{R}}(x) = \exp(r \cdot x).$
- (6) If  $r \neq 0$ , then  $\int_A ((\text{the function exp}) \cdot (r\square+0))(x)dx = \frac{1}{r} \cdot \exp(r \cdot \sup A) - \frac{1}{r} \cdot \exp(r \cdot \inf A).$
- (7)  $\int_A ((\text{the function sin}) \cdot (2\square+0))(x)dx = (-\frac{1}{2}) \cdot \cos(2 \cdot \sup A) - (-\frac{1}{2}) \cdot \cos(2 \cdot \inf A).$
- (8) Suppose  $n \neq 0$ . Then  $(-\frac{1}{n}) ((\text{the function cos}) \cdot (n\square+0))$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $((-\frac{1}{n}) ((\text{the function cos}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = \sin(n \cdot x).$
- (9) If  $n \neq 0$ , then  $\int_A ((\text{the function sin}) \cdot (n\square+0))(x)dx = (-\frac{1}{n}) \cdot \cos(n \cdot \sup A) - (-\frac{1}{n}) \cdot \cos(n \cdot \inf A).$
- (10)  $\frac{1}{2} ((\text{the function sin}) \cdot (2\square+0))$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(\frac{1}{2} ((\text{the function sin}) \cdot (2\square+0)))'_{\mathbb{R}}(x) = \cos(2 \cdot x).$
- (11)  $\int_A ((\text{the function cos}) \cdot (2\square+0))(x)dx = \frac{1}{2} \cdot \sin(2 \cdot \sup A) - \frac{1}{2} \cdot \sin(2 \cdot \inf A).$
- (12) Suppose  $n \neq 0$ . Then  $\frac{1}{n} ((\text{the function sin}) \cdot (n\square+0))$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(\frac{1}{n} ((\text{the function sin}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = \cos(n \cdot x).$
- (13) If  $n \neq 0$ , then  $\int_A ((\text{the function cos}) \cdot (n\square+0))(x)dx = \frac{1}{n} \cdot \sin(n \cdot \sup A) - \frac{1}{n} \cdot \sin(n \cdot \inf A).$
- (14) If  $A \subseteq Z$ , then  $\int_A (\text{id}_Z (\text{the function sin}))(x)dx = ((-\sup A) \cdot \cos \sup A + \sin \sup A) - ((-\inf A) \cdot \cos \inf A + \sin \inf A).$
- (15) If  $A \subseteq Z$ , then  $\int_A (\text{id}_Z (\text{the function cos}))(x)dx = (\sup A \cdot \sin \sup A + \cos \sup A) - (\inf A \cdot \sin \inf A + \cos \inf A).$

- (16)  $\text{id}_Z$  (the function  $\cos$ ) is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(\text{id}_Z (\text{the function } \cos))'_{\uparrow Z}(x) = \cos x - x \cdot \sin x$ .
- (17)(i)  $-\text{the function } \sin + \text{id}_Z$  (the function  $\cos$ ) is differentiable on  $Z$ , and  
(ii) for every  $x$  such that  $x \in Z$  holds  $(-\text{the function } \sin + \text{id}_Z (\text{the function } \cos))'_{\uparrow Z}(x) = -x \cdot \sin x$ .
- (18) If  $A \subseteq Z$ , then  $\int_A ((-\text{id}_Z) (\text{the function } \sin))(x)dx = (-\sin \sup A + \sup A \cdot \cos \sup A) - (-\sin \inf A + \inf A \cdot \cos \inf A)$ .
- (19)(i)  $-\text{the function } \cos - \text{id}_Z$  (the function  $\sin$ ) is differentiable on  $Z$ , and  
(ii) for every  $x$  such that  $x \in Z$  holds  $(-\text{the function } \cos - \text{id}_Z (\text{the function } \sin))'_{\uparrow Z}(x) = -x \cdot \cos x$ .
- (20) If  $A \subseteq Z$ , then  $\int_A ((-\text{id}_Z) (\text{the function } \cos))(x)dx = -\cos \sup A - \sup A \cdot \sin \sup A - (-\cos \inf A - \inf A \cdot \sin \inf A)$ .
- (21) If  $A \subseteq Z$ , then  $\int_A ((\text{the function } \sin) + \text{id}_Z (\text{the function } \cos))(x)dx = \sup A \cdot \sin \sup A - \inf A \cdot \sin \inf A$ .
- (22) If  $A \subseteq Z$ , then  $\int_A (-\text{the function } \cos + \text{id}_Z (\text{the function } \sin))(x)dx = (-\sup A) \cdot \cos \sup A - (-\inf A) \cdot \cos \inf A$ .
- (23)  $\int_A ((1 \square + 0) (\text{the function } \exp))(x)dx = \exp(\sup A - 1) - \exp(\inf A - 1)$ .
- (24)  $\frac{1}{n+1} (\square^{n+1})$  is differentiable on  $\mathbb{R}$  and for every  $x$  holds  $(\frac{1}{n+1} (\square^{n+1}))'_{\uparrow \mathbb{R}}(x) = x^n$ .
- (25)  $\int_A (\square^n)(x)dx = \frac{1}{n+1} \cdot (\sup A)^{n+1} - \frac{1}{n+1} \cdot (\inf A)^{n+1}$ .
- (26) For all partial functions  $f, g$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every non empty subset  $C$  of  $\mathbb{R}$  holds  $(f - g) \uparrow C = f \uparrow C - g \uparrow C$ .
- (27) For all partial functions  $f_1, f_2, g$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every non empty subset  $C$  of  $\mathbb{R}$  holds  $((f_1 + f_2) \uparrow C) (g \uparrow C) = (f_1 g + f_2 g) \uparrow C$ .
- (28) For all partial functions  $f_1, f_2, g$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every non empty subset  $C$  of  $\mathbb{R}$  holds  $((f_1 - f_2) \uparrow C) (g \uparrow C) = (f_1 g - f_2 g) \uparrow C$ .
- (29) For all partial functions  $f_1, f_2, g$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every non empty subset  $C$  of  $\mathbb{R}$  holds  $((f_1 f_2) \uparrow C) (g \uparrow C) = (f_1 \uparrow C) ((f_2 g) \uparrow C)$ .

Let  $A$  be a closed-interval subset of  $\mathbb{R}$  and let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ . The functor  $\langle f, g \rangle_A$  yielding a real number is defined by:

$$\text{(Def. 1)} \quad \langle f, g \rangle_A = \int_A (f g)(x)dx.$$

The following propositions are true:

- (30) For all partial functions  $f, g$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every closed-interval subset  $A$  of  $\mathbb{R}$  holds  $\langle f, g \rangle_A = \langle g, f \rangle_A$ .
- (31) Let  $f_1, f_2, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose that
- (i)  $(f_1 g) \upharpoonright A$  is total,
  - (ii)  $(f_2 g) \upharpoonright A$  is total,
  - (iii)  $(f_1 g) \upharpoonright A$  is bounded,
  - (iv)  $f_1 g$  is integrable on  $A$ ,
  - (v)  $(f_2 g) \upharpoonright A$  is bounded, and
  - (vi)  $f_2 g$  is integrable on  $A$ .
- Then  $\langle f_1 + f_2, g \rangle_A = \langle (f_1), g \rangle_A + \langle (f_2), g \rangle_A$ .
- (32) Let  $f_1, f_2, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose that
- (i)  $(f_1 g) \upharpoonright A$  is total,
  - (ii)  $(f_2 g) \upharpoonright A$  is total,
  - (iii)  $(f_1 g) \upharpoonright A$  is bounded,
  - (iv)  $f_1 g$  is integrable on  $A$ ,
  - (v)  $(f_2 g) \upharpoonright A$  is bounded, and
  - (vi)  $f_2 g$  is integrable on  $A$ .
- Then  $\langle f_1 - f_2, g \rangle_A = \langle (f_1), g \rangle_A - \langle (f_2), g \rangle_A$ .
- (33) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose  $(f g) \upharpoonright A$  is bounded and  $f g$  is integrable on  $A$  and  $A \subseteq \text{dom}(f g)$ . Then  $\langle -f, g \rangle_A = -\langle f, g \rangle_A$ .
- (34) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose  $(f g) \upharpoonright A$  is bounded and  $f g$  is integrable on  $A$  and  $A \subseteq \text{dom}(f g)$ . Then  $\langle r f, g \rangle_A = r \cdot \langle f, g \rangle_A$ .
- (35) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose  $(f g) \upharpoonright A$  is bounded and  $f g$  is integrable on  $A$  and  $A \subseteq \text{dom}(f g)$ . Then  $\langle r f, p g \rangle_A = r \cdot p \cdot \langle f, g \rangle_A$ .
- (36) For all partial functions  $f, g, h$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every closed-interval subset  $A$  of  $\mathbb{R}$  holds  $\langle f g, h \rangle_A = \langle f, g h \rangle_A$ .
- (37) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose that  $(f f) \upharpoonright A$  is total and  $(f g) \upharpoonright A$  is total and  $(g g) \upharpoonright A$  is total and  $(f f) \upharpoonright A$  is bounded and  $(f g) \upharpoonright A$  is bounded and  $(g g) \upharpoonright A$  is bounded and  $f f$  is integrable on  $A$  and  $f g$  is integrable on  $A$  and  $g g$  is integrable on  $A$ . Then  $\langle f + g, f + g \rangle_A = \langle f, f \rangle_A + 2 \cdot \langle f, g \rangle_A + \langle g, g \rangle_A$ .

Let  $A$  be a closed-interval subset of  $\mathbb{R}$  and let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ . We say that  $f$  is orthogonal with  $g$  in  $A$  if and only if:

(Def. 2)  $\langle f, g \rangle_A = 0$ .

The following propositions are true:

- (38) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose that  $(f f) \upharpoonright A$  is total and  $(f g) \upharpoonright A$  is total and  $(g g) \upharpoonright A$  is total and  $(f f) \upharpoonright A$  is bounded and  $(f g) \upharpoonright A$  is bounded and  $(g g) \upharpoonright A$  is bounded and  $f f$  is integrable on  $A$  and  $f g$  is integrable on  $A$  and  $g g$  is integrable on  $A$  and  $f$  is orthogonal with  $g$  in  $A$ . Then  $\langle f + g, f + g \rangle_A = \langle f, f \rangle_A + \langle g, g \rangle_A$ .
- (39) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose  $(f f) \upharpoonright A$  is total and  $(f f) \upharpoonright A$  is bounded and  $f f$  is integrable on  $A$  and for every  $x$  such that  $x \in A$  holds  $((f f) \upharpoonright A)(x) \geq 0$ . Then  $\langle f, f \rangle_A \geq 0$ .
- (40) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[0, \pi]$ .
- (41) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[0, \pi \cdot 2]$ .
- (42) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$ .
- (43) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$ .
- (44) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[-\pi, \pi]$ .
- (45) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .
- (46) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[-2 \cdot \pi, 2 \cdot \pi]$ .
- (47) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[-2 \cdot n \cdot \pi, 2 \cdot n \cdot \pi]$ .
- (48) The function  $\sin$  is orthogonal with the function  $\cos$  in  $[x - 2 \cdot n \cdot \pi, x + 2 \cdot n \cdot \pi]$ .

Let  $A$  be a closed-interval subset of  $\mathbb{R}$  and let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . The functor  $\|f\|_A$  yields a real number and is defined by:

(Def. 3)  $\|f\|_A = \sqrt{\langle f, f \rangle_A}$ .

Next we state three propositions:

- (49) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose  $(f f) \upharpoonright A$  is total and  $(f f) \upharpoonright A$  is bounded and  $f f$  is integrable on  $A$  and for every  $x$  such that  $x \in A$  holds  $((f f) \upharpoonright A)(x) \geq 0$ . Then  $0 \leq \|f\|_A$ .
- (50) For every partial function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  and for every closed-interval subset  $A$  of  $\mathbb{R}$  holds  $\|1 f\|_A = \|f\|_A$ .
- (51) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $A$  be a closed-interval subset of  $\mathbb{R}$ . Suppose that  $(f f) \upharpoonright A$  is total and  $(f g) \upharpoonright A$  is total and  $(g g) \upharpoonright A$  is total and  $(f f) \upharpoonright A$  is bounded and  $(f g) \upharpoonright A$  is bounded and  $(g g) \upharpoonright A$  is bounded and  $f f$  is integrable on  $A$  and  $f g$  is integrable on  $A$  and  $g g$  is integrable on  $A$  and  $f$  is orthogonal with  $g$  in  $A$  and for every  $x$  such that  $x \in A$  holds  $((f f) \upharpoonright A)(x) \geq 0$  and for every  $x$  such that  $x \in A$  holds  $((g g) \upharpoonright A)(x) \geq 0$ . Then  $(\|f + g\|_A)^2 = (\|f\|_A)^2 + (\|g\|_A)^2$ .

For simplicity, we follow the rules:  $a, b, x$  are real numbers,  $n$  is an element of  $\mathbb{N}$ ,  $A$  is a closed-interval subset of  $\mathbb{R}$ ,  $f, f_1, f_2$  are partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ , and  $Z$  is an open subset of  $\mathbb{R}$ .

Next we state several propositions:

(52) If  $-a \notin A$ , then  $\frac{1}{1 \square + a} \upharpoonright A$  is continuous.

(53) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a + x$  and  $f(x) \neq 0$ ,
- (iii)  $Z = \text{dom } f$ ,
- (iv)  $\text{dom } f = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = -\frac{1}{(a+x)^2}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = f(\sup A)^{-1} - f(\inf A)^{-1}.$$

(54) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a + x$  and  $f(x) \neq 0$ ,
- (iii)  $\text{dom}((-1) \frac{1}{f}) = Z$ ,
- (iv)  $\text{dom}((-1) \frac{1}{f}) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{(a+x)^2}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = -f(\sup A)^{-1} + f(\inf A)^{-1}.$$

(55) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a - x$  and  $f(x) \neq 0$ ,
- (iii)  $\text{dom } f = Z$ ,
- (iv)  $\text{dom } f = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{(a-x)^2}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = f(\sup A)^{-1} - f(\inf A)^{-1}.$$

(56) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a + x$  and  $f(x) > 0$ ,
- (iii)  $\text{dom}((\text{the function } \ln) \cdot f) = Z$ ,
- (iv)  $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{a+x}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln(a + \sup A) - \ln(a + \inf A).$$

Next we state a number of propositions:

(57) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = x - a$  and  $f(x) > 0$ ,
- (iii)  $\text{dom}((\text{the function } \ln) \cdot f) = Z$ ,
- (iv)  $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{x-a}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln f(\sup A) - \ln f(\inf A).$$

(58) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a - x$  and  $f(x) > 0$ ,
- (iii)  $\text{dom}(-(\text{the function } \ln) \cdot f) = Z$ ,
- (iv)  $\text{dom}(-(\text{the function } \ln) \cdot f) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{a-x}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = -\ln(a - \sup A) + \ln(a - \inf A).$$

(59) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a + x$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z - a f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x}{a+x}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x) dx = \sup A - a \cdot f(\sup A) - (\inf A - a \cdot f(\inf A))$ .

(60) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a + x$  and  $f_1(x) > 0$  and  $\text{dom}((2 \cdot a) f - \text{id}_Z) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{a-x}{a+x}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x) dx = 2 \cdot a \cdot f(\sup A) - \sup A - (2 \cdot a \cdot f(\inf A) - \inf A)$ .

(61) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x + a$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z - (2 \cdot a) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x-a}{x+a}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x) dx = \sup A - 2 \cdot a \cdot f(\sup A) - (\inf A - 2 \cdot a \cdot f(\inf A))$ .

(62) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x - a$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z + (2 \cdot a) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x+a}{x-a}$  and  $f_2 \upharpoonright A$

is continuous. Then  $\int_A f_2(x)dx = (\sup A + 2 \cdot a \cdot f(\sup A)) - (\inf A + 2 \cdot a \cdot f(\inf A))$ .

(63) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x + b$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z + (a - b) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x+a}{x+b}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x)dx = (\sup A + (a - b) \cdot f(\sup A)) - (\inf A + (a - b) \cdot f(\inf A))$ .

(64) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x - b$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z + (a + b) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x+a}{x-b}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x)dx = (\sup A + (a + b) \cdot f(\sup A)) - (\inf A + (a + b) \cdot f(\inf A))$ .

(65) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x + b$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z - (a + b) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x-a}{x+b}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x)dx = \sup A - (a + b) \cdot f(\sup A) - (\inf A - (a + b) \cdot f(\inf A))$ .

(66) Suppose that  $A \subseteq Z$  and  $f = (\text{the function } \ln) \cdot f_1$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x - b$  and  $f_1(x) > 0$  and  $\text{dom}(\text{id}_Z + (b - a) f) = Z = \text{dom } f_2$  and for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{x-a}{x-b}$  and  $f_2 \upharpoonright A$  is continuous. Then  $\int_A f_2(x)dx = (\sup A + (b - a) \cdot f(\sup A)) - (\inf A + (b - a) \cdot f(\inf A))$ .

(67) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = x$  and  $f(x) > 0$ ,
- (iii)  $\text{dom}((\text{the function } \ln) \cdot f) = Z$ ,
- (iv)  $\text{dom}((\text{the function } \ln) \cdot f) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{1}{x}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

Then  $\int_A f_2(x)dx = \ln \sup A - \ln \inf A$ .

(68) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $x > 0$ ,
- (iii)  $\text{dom}((\text{the function } \ln) \cdot (\square^n)) = Z$ ,



- (iv)  $\text{dom}((\text{the function } \ln) \cdot (\square^n)) = \text{dom } f_2,$
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = \frac{n}{x},$  and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = \ln((\sup A)^n) - \ln((\inf A)^n).$$

(69) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = x,$
- (iii)  $\text{dom}((\text{the function } \ln) \cdot \frac{1}{f}) = Z,$
- (iv)  $\text{dom}((\text{the function } \ln) \cdot \frac{1}{f}) = \text{dom } f_2,$
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = -\frac{1}{x},$  and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = -\ln \sup A + \ln \inf A.$$

(70) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a + x$  and  $f(x) > 0,$
- (iii)  $\text{dom}(\frac{2}{3} f^{\frac{3}{2}}) = Z,$
- (iv)  $\text{dom}(\frac{2}{3} f^{\frac{3}{2}}) = \text{dom } f_2,$
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = (a + x)^{\frac{1}{2}},$  and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = \frac{2}{3} \cdot (a + \sup A)^{\frac{3}{2}} - \frac{2}{3} \cdot (a + \inf A)^{\frac{3}{2}}.$$

(71) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a - x$  and  $f(x) > 0,$
- (iii)  $\text{dom}((-\frac{2}{3}) f^{\frac{3}{2}}) = Z,$
- (iv)  $\text{dom}((-\frac{2}{3}) f^{\frac{3}{2}}) = \text{dom } f_2,$
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = (a - x)^{\frac{1}{2}},$  and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x) dx = -\frac{2}{3} \cdot (a - \sup A)^{\frac{3}{2}} + \frac{2}{3} \cdot (a - \inf A)^{\frac{3}{2}}.$$

(72) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a + x$  and  $f(x) > 0,$
- (iii)  $\text{dom}(2 f^{\frac{1}{2}}) = Z,$
- (iv)  $\text{dom}(2 f^{\frac{1}{2}}) = \text{dom } f_2,$
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = (a + x)^{-\frac{1}{2}},$  and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x)dx = 2 \cdot (a + \sup A)^{\frac{1}{2}} - 2 \cdot (a + \inf A)^{\frac{1}{2}}.$$

(73) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = a - x$  and  $f(x) > 0$ ,
- (iii)  $\text{dom}((-2) f^{\frac{1}{2}}) = Z$ ,
- (iv)  $\text{dom}((-2) f^{\frac{1}{2}}) = \text{dom } f_2$ ,
- (v) for every  $x$  such that  $x \in Z$  holds  $f_2(x) = (a - x)^{-\frac{1}{2}}$ , and
- (vi)  $f_2 \upharpoonright A$  is continuous.

$$\text{Then } \int_A f_2(x)dx = -2 \cdot (a - \sup A)^{\frac{1}{2}} + 2 \cdot (a - \inf A)^{\frac{1}{2}}.$$

(74) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii)  $\text{dom}((-id_Z)(\text{the function cos}) + \text{the function sin}) = Z$ ,
- (iii) for every  $x$  such that  $x \in Z$  holds  $f(x) = x \cdot \sin x$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = (-\sup A \cdot \cos \sup A + \sin \sup A) - (-\inf A \cdot \cos \inf A + \sin \inf A).$$

(75) Suppose  $A \subseteq Z$  and  $\text{dom}(\text{the function sec}) = Z$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{\sin x}{(\cos x)^2}$  and  $Z = \text{dom } f$  and  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = \sec \sup A - \sec \inf A.$$

(76) Suppose  $Z \subseteq \text{dom}(-\text{the function cosec})$ . Then  $-\text{the function cosec}$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(-\text{the function cosec})' \upharpoonright_Z(x) = \frac{\cos x}{(\sin x)^2}$ .

(77) Suppose  $A \subseteq Z$  and  $\text{dom}(-\text{the function cosec}) = Z$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{\cos x}{(\sin x)^2}$  and  $Z = \text{dom } f$  and  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = -\text{cosec } \sup A + \text{cosec } \inf A.$$

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