

Several Integrability Formulas of Special Functions. Part II

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Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including the trigonometric function, the hyperbolic function and the polynomial function [3].

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The articles [10], [23], [19], [21], [22], [1], [8], [15], [9], [2], [4], [17], [5], [13], [16], [14], [18], [7], [12], [20], [6], and [11] provide the terminology and notation for this paper.

1. DIFFERENTIATION FORMULAS

For simplicity, we adopt the following rules: r, x, a, b denote real numbers, n, m denote elements of \mathbb{N} , A denotes a closed-interval subset of \mathbb{R} , and Z denotes an open subset of \mathbb{R} .

One can prove the following propositions:

- (1)(i) $(\frac{1}{2}\square+0) - \frac{1}{4} ((\text{the function } \sin) \cdot (2\square+0))$ is differentiable on \mathbb{R} , and
- (ii) for every x holds $((\frac{1}{2}\square+0) - \frac{1}{4} ((\text{the function } \sin) \cdot (2\square+0)))'_{\mathbb{R}}(x) = (\sin x)^2$.

- (2)(i) $(\frac{1}{2}\square+0) + \frac{1}{4}((\text{the function sin}) \cdot (2\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $((\frac{1}{2}\square+0) + \frac{1}{4}((\text{the function sin}) \cdot (2\square+0)))'_{\mathbb{R}}(x) = (\cos x)^2$.
- (3) $\frac{1}{n+1}((\square^{n+1}) \cdot (\text{the function sin}))$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{n+1}(\text{the function sin})^{n+1})'_{\mathbb{R}}(x) = (\sin x)^n \cdot \cos x$.
- (4)(i) $(-\frac{1}{n+1})((\square^{n+1}) \cdot (\text{the function cos}))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $((-\frac{1}{n+1})(\text{the function cos})^{n+1})'_{\mathbb{R}}(x) = (\cos x)^n \cdot \sin x$.
- (5) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then
(i) $\frac{1}{2 \cdot (m+n)}((\text{the function sin}) \cdot ((m+n)\square+0)) + \frac{1}{2 \cdot (m-n)}((\text{the function sin}) \cdot ((m-n)\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $(\frac{1}{2 \cdot (m+n)}((\text{the function sin}) \cdot ((m+n)\square+0)) + \frac{1}{2 \cdot (m-n)}((\text{the function sin}) \cdot ((m-n)\square+0)))'_{\mathbb{R}}(x) = \cos(m \cdot x) \cdot \cos(n \cdot x)$.
- (6) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then
(i) $\frac{1}{2 \cdot (m-n)}((\text{the function sin}) \cdot ((m-n)\square+0)) - \frac{1}{2 \cdot (m+n)}((\text{the function sin}) \cdot ((m+n)\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $(\frac{1}{2 \cdot (m-n)}((\text{the function sin}) \cdot ((m-n)\square+0)) - \frac{1}{2 \cdot (m+n)}((\text{the function sin}) \cdot ((m+n)\square+0)))'_{\mathbb{R}}(x) = \sin(m \cdot x) \cdot \sin(n \cdot x)$.
- (7) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then
(i) $-\frac{1}{2 \cdot (m+n)}((\text{the function cos}) \cdot ((m+n)\square+0)) - \frac{1}{2 \cdot (m-n)}((\text{the function cos}) \cdot ((m-n)\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $(-\frac{1}{2 \cdot (m+n)}((\text{the function cos}) \cdot ((m+n)\square+0)) - \frac{1}{2 \cdot (m-n)}((\text{the function cos}) \cdot ((m-n)\square+0)))'_{\mathbb{R}}(x) = \sin(m \cdot x) \cdot \cos(n \cdot x)$.
- (8) Suppose $n \neq 0$. Then
(i) $\frac{1}{n^2}((\text{the function sin}) \cdot (n\square+0)) - (\frac{1}{n}\square+0)((\text{the function cos}) \cdot (n\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $(\frac{1}{n^2}((\text{the function sin}) \cdot (n\square+0)) - (\frac{1}{n}\square+0)((\text{the function cos}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = x \cdot \sin(n \cdot x)$.
- (9) Suppose $n \neq 0$. Then
(i) $\frac{1}{n^2}((\text{the function cos}) \cdot (n\square+0)) + (\frac{1}{n}\square+0)((\text{the function sin}) \cdot (n\square+0))$ is differentiable on \mathbb{R} , and
(ii) for every x holds $(\frac{1}{n^2}((\text{the function cos}) \cdot (n\square+0)) + (\frac{1}{n}\square+0)((\text{the function sin}) \cdot (n\square+0)))'_{\mathbb{R}}(x) = x \cdot \cos(n \cdot x)$.
- (10)(i) $(1\square+0)(\text{the function cosh}) - \text{the function sinh}$ is differentiable on \mathbb{R} , and
(ii) for every x holds $((1\square+0)(\text{the function cosh}) - \text{the function sinh})'_{\mathbb{R}}(x) = x \cdot \sinh x$.
- (11)(i) $(1\square+0)(\text{the function sinh}) - \text{the function cosh}$ is differentiable on \mathbb{R} , and

- (ii) for every x holds $((1 \square + 0)$ (the function \sinh)—the function \cosh)' $_{\mathbb{R}}(x) = x \cdot \cosh x$.
- (12) If $a \cdot (n + 1) \neq 0$, then $\frac{1}{a \cdot (n+1)} (a \square + b)^{n+1}$ is differentiable on \mathbb{R} and for every x holds $(\frac{1}{a \cdot (n+1)} (a \square + b)^{n+1})'_{\mathbb{R}}(x) = (a \cdot x + b)^n$.

2. INTEGRABILITY FORMULAS

Next we state a number of propositions:

- (13) $\int_A (\text{the function } \sin)^2(x) dx = \frac{1}{2} \cdot \sup A - \frac{1}{4} \cdot \sin(2 \cdot \sup A) - (\frac{1}{2} \cdot \inf A - \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.
- (14) $\int_{[0, \pi]} (\text{the function } \sin)^2(x) dx = \frac{\pi}{2}$.
- (15) $\int_{[0, 2 \cdot \pi]} (\text{the function } \sin)^2(x) dx = \pi$.
- (16) $\int_A (\text{the function } \cos)^2(x) dx = (\frac{1}{2} \cdot \sup A + \frac{1}{4} \cdot \sin(2 \cdot \sup A)) - (\frac{1}{2} \cdot \inf A + \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.
- (17) $\int_{[0, \pi]} (\text{the function } \cos)^2(x) dx = \frac{\pi}{2}$.
- (18) $\int_{[0, 2 \cdot \pi]} (\text{the function } \cos)^2(x) dx = \pi$.
- (19) $\int_A ((\text{the function } \sin)^n (\text{the function } \cos))(x) dx = \frac{1}{n+1} \cdot (\sin \sup A)^{n+1} - \frac{1}{n+1} \cdot (\sin \inf A)^{n+1}$.
- (20) $\int_{[0, \pi]} ((\text{the function } \sin)^n (\text{the function } \cos))(x) dx = 0$.
- (21) $\int_{[0, 2 \cdot \pi]} ((\text{the function } \sin)^n (\text{the function } \cos))(x) dx = 0$.
- (22) $\int_A ((\text{the function } \cos)^n (\text{the function } \sin))(x) dx = (-\frac{1}{n+1}) \cdot (\cos \sup A)^{n+1} - (-\frac{1}{n+1}) \cdot (\cos \inf A)^{n+1}$.

- (23) $\int_{[0, 2\cdot\pi]} ((\text{the function } \cos)^n (\text{the function } \sin))(x)dx = 0.$
- (24) $\int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} ((\text{the function } \cos)^n (\text{the function } \sin))(x)dx = 0.$
- (25) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

$$\int_A (((\text{the function } \cos) \cdot (m\Box+0)) ((\text{the function } \cos) \cdot (n\Box+0)))(x)dx =$$

$$\left(\frac{1}{2 \cdot (m+n)} \cdot \sin((m+n) \cdot \sup A) + \frac{1}{2 \cdot (m-n)} \cdot \sin((m-n) \cdot \sup A)\right) -$$

$$\left(\frac{1}{2 \cdot (m+n)} \cdot \sin((m+n) \cdot \inf A) + \frac{1}{2 \cdot (m-n)} \cdot \sin((m-n) \cdot \inf A)\right).$$
- (26) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

$$\int_A (((\text{the function } \sin) \cdot (m\Box+0)) ((\text{the function } \sin) \cdot (n\Box+0)))(x)dx =$$

$$\frac{1}{2 \cdot (m-n)} \cdot \sin((m-n) \cdot \sup A) - \frac{1}{2 \cdot (m+n)} \cdot \sin((m+n) \cdot \sup A) -$$

$$\left(\frac{1}{2 \cdot (m-n)} \cdot \sin((m-n) \cdot \inf A) - \frac{1}{2 \cdot (m+n)} \cdot \sin((m+n) \cdot \inf A)\right).$$
- (27) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

$$\int_A (((\text{the function } \sin) \cdot (m\Box+0)) ((\text{the function } \cos) \cdot (n\Box+0)))(x)dx =$$

$$-\frac{1}{2 \cdot (m+n)} \cdot \cos((m+n) \cdot \sup A) - \frac{1}{2 \cdot (m-n)} \cdot \cos((m-n) \cdot \sup A) -$$

$$\left(-\frac{1}{2 \cdot (m+n)} \cdot \cos((m+n) \cdot \inf A) - \frac{1}{2 \cdot (m-n)} \cdot \cos((m-n) \cdot \inf A)\right).$$
- (28) If $n \neq 0$, then $\int_A ((1\Box+0) ((\text{the function } \sin) \cdot (n\Box+0)))(x)dx = \frac{1}{n^2} \cdot$

$$\sin(n \cdot \sup A) - \frac{1}{n} \cdot \sup A \cdot \cos(n \cdot \sup A) - \left(\frac{1}{n^2} \cdot \sin(n \cdot \inf A) - \frac{1}{n} \cdot \inf A \cdot \right.$$

$$\left. \cos(n \cdot \inf A)\right).$$
- (29) If $n \neq 0$, then $\int_A ((1\Box+0) ((\text{the function } \cos) \cdot (n\Box+0)))(x)dx = \left(\frac{1}{n^2} \cdot \right.$

$$\cos(n \cdot \sup A) + \frac{1}{n} \cdot \sup A \cdot \sin(n \cdot \sup A)\right) - \left(\frac{1}{n^2} \cdot \cos(n \cdot \inf A) + \frac{1}{n} \cdot \inf A \cdot \right.$$

$$\left. \sin(n \cdot \inf A)\right).$$
- (30) $\int_A ((1\Box+0) (\text{the function } \sinh))(x)dx = \sup A \cdot \cosh \sup A - \sinh \sup A -$

$$(\inf A \cdot \cosh \inf A - \sinh \inf A).$$
- (31) $\int_A ((1\Box+0) (\text{the function } \cosh))(x)dx = \sup A \cdot \sinh \sup A - \cosh \sup A -$

$$(\inf A \cdot \sinh \inf A - \cosh \inf A).$$

$$(32) \quad \text{If } a \cdot (n+1) \neq 0, \text{ then } \int_A (a \square + b)^n(x) dx = \frac{1}{a \cdot (n+1)} \cdot (a \cdot \sup A + b)^{n+1} - \frac{1}{a \cdot (n+1)} \cdot (a \cdot \inf A + b)^{n+1}.$$

3. ADDENDA

In the sequel f, f_1, f_2, f_3, g are partial functions from \mathbb{R} to \mathbb{R} .

The following propositions are true:

$$(33) \quad \text{If } Z \subseteq \text{dom}(\frac{1}{2} f) \text{ and } f = \square^2, \text{ then } \frac{1}{2} f \text{ is differentiable on } Z \text{ and for every } x \text{ such that } x \in Z \text{ holds } (\frac{1}{2} f)'_{\downarrow Z}(x) = x.$$

$$(34) \quad \text{If } A \subseteq Z = \text{dom}(\frac{1}{2} (\square^2)), \text{ then } \int_A \text{id}_Z(x) dx = \frac{1}{2} \cdot (\sup A)^2 - \frac{1}{2} \cdot (\inf A)^2.$$

$$(35) \quad \text{Suppose } A \subseteq Z \text{ and for every } x \text{ such that } x \in Z \text{ holds } g(x) = x \text{ and } g(x) \neq 0 \text{ and } f(x) = -\frac{1}{x^2} \text{ and } Z = \text{dom } g \text{ and } \text{dom } f = Z \text{ and } f \upharpoonright A \text{ is continuous. Then } \int_A f(x) dx = (\sup A)^{-1} - (\inf A)^{-1}.$$

(36) Suppose that

$$(i) \quad A \subseteq Z,$$

$$(ii) \quad f_1 = \square^2,$$

$$(iii) \quad \text{for every } x \text{ such that } x \in Z \text{ holds } f_2(x) = 1 \text{ and } x \neq 0 \text{ and } f(x) = \frac{2 \cdot x}{(1+x^2)^2},$$

$$(iv) \quad \text{dom}(\frac{f_1}{f_2+f_1}) = Z,$$

$$(v) \quad Z = \text{dom } f, \text{ and}$$

(vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = (\frac{f_1}{f_2+f_1})(\sup A) - (\frac{f_1}{f_2+f_1})(\inf A).$$

(37) Suppose $Z \subseteq \text{dom}(\text{(the function tan)} + \text{(the function sec)})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) $\text{(the function tan)} + \text{(the function sec)}$ is differentiable on Z , and

$$(ii) \quad \text{for every } x \text{ such that } x \in Z \text{ holds } ((\text{the function tan}) + \text{(the function sec)})'_{\downarrow Z}(x) = \frac{1}{1 - \sin x}.$$

(38) Suppose that

$$(i) \quad A \subseteq Z,$$

$$(ii) \quad \text{for every } x \text{ such that } x \in Z \text{ holds } 1 + \sin x \neq 0 \text{ and } 1 - \sin x \neq 0 \text{ and } f(x) = \frac{1}{1 - \sin x},$$

$$(iii) \quad \text{dom}(\text{(the function tan)} + \text{(the function sec)}) = Z,$$

$$(iv) \quad Z = \text{dom } f, \text{ and}$$

(v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = (\tan \sup A + \sec \sup A) - (\tan \inf A + \sec \inf A)$.

(39) Suppose $Z \subseteq \text{dom}(\text{(the function tan)} - \text{(the function sec)})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

- (i) $\text{(the function tan)} - \text{(the function sec)}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $\text{(the function tan)} - \text{(the function sec)} \Big|_Z(x) = \frac{1}{1 + \sin x}$.

(40) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{1}{1 + \sin x}$,
- (iii) $\text{dom}(\text{(the function tan)} - \text{(the function sec)}) = Z$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = \tan \sup A - \sec \sup A - (\tan \inf A - \sec \inf A)$.

(41) Suppose $Z \subseteq \text{dom}(-\text{the function cot} + \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

- (i) $-\text{the function cot} + \text{the function cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(-\text{the function cot} + \text{the function cosec}) \Big|_Z(x) = \frac{1}{1 + \cos x}$.

(42) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1 + \cos x}$,
- (iii) $\text{dom}(-\text{the function cot} + \text{the function cosec}) = Z$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = (-\cot \sup A + \text{cosec} \sup A) - (-\cot \inf A + \text{cosec} \inf A)$.

(43) Suppose $Z \subseteq \text{dom}(-\text{the function cot} - \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

- (i) $-\text{the function cot} - \text{the function cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(-\text{the function cot} - \text{the function cosec}) \Big|_Z(x) = \frac{1}{1 - \cos x}$.

(44) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1 - \cos x}$,
- (iii) $\text{dom}(-\text{the function cot} - \text{the function cosec}) = Z$,
- (iv) $Z = \text{dom } f$, and

(v) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = -\cot \sup A - \operatorname{cosec} \sup A - (-\cot \inf A - \operatorname{cosec} \inf A).$$

(45) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{1+x^2}$,
- (iv) $\operatorname{dom}(\text{the function } \arctan) = Z$,
- (v) $Z = \operatorname{dom} f$, and
- (vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = \arctan \sup A - \arctan \inf A.$$

(46) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{r}{1+x^2}$,
- (iv) $\operatorname{dom}(r \text{ the function } \arctan) = Z$,
- (v) $Z = \operatorname{dom} f$, and
- (vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = r \cdot \arctan \sup A - r \cdot \arctan \inf A.$$

(47) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{1+x^2}$,
- (iv) $\operatorname{dom}(\text{the function } \operatorname{arccot}) = Z$,
- (v) $Z = \operatorname{dom} f$, and
- (vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = \operatorname{arccot} \sup A - \operatorname{arccot} \inf A.$$

(48) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = -\frac{r}{1+x^2}$,
- (iv) $\operatorname{dom}(r \text{ the function } \operatorname{arccot}) = Z$,
- (v) $Z = \operatorname{dom} f$, and
- (vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x) dx = r \cdot \operatorname{arccot} \sup A - r \cdot \operatorname{arccot} \inf A.$$

(49) Suppose $Z \subseteq \operatorname{dom}((\operatorname{id}_Z + \text{the function } \cot) - \text{the function } \operatorname{cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

- (i) $(\text{id}_Z + \text{the function cot})$ –the function cosec is differentiable on Z , and
(ii) for every x such that $x \in Z$ holds $(\text{id}_Z + \text{the function cot})$ –the function cosec $'_{\upharpoonright Z}(x) = \frac{\cos x}{1 + \cos x}$.
- (50) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{1 + \cos x}$,
(iii) $\text{dom}((\text{id}_Z + \text{the function cot})\text{–the function cosec}) = Z$,
(iv) $Z = \text{dom } f$, and
(v) $f \upharpoonright A$ is continuous.
Then $\int_A f(x)dx = (\sup A + \cot \sup A) - \text{cosec } \sup A - ((\inf A + \cot \inf A) - \text{cosec } \inf A)$.
- (51) Suppose $Z \subseteq \text{dom}(\text{id}_Z + \text{the function cot} + \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then
(i) $\text{id}_Z + \text{the function cot} + \text{the function cosec}$ is differentiable on Z , and
(ii) for every x such that $x \in Z$ holds $(\text{id}_Z + \text{the function cot} + \text{the function cosec})'_{\upharpoonright Z}(x) = \frac{\cos x}{\cos x - 1}$.
- (52) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{\cos x - 1}$,
(iii) $\text{dom}(\text{id}_Z + \text{the function cot} + \text{the function cosec}) = Z$,
(iv) $Z = \text{dom } f$, and
(v) $f \upharpoonright A$ is continuous.
Then $\int_A f(x)dx = (\sup A + \cot \sup A + \text{cosec } \sup A) - (\inf A + \cot \inf A + \text{cosec } \inf A)$.
- (53) Suppose $Z \subseteq \text{dom}((\text{id}_Z - \text{the function tan}) + \text{the function sec})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then
(i) $(\text{id}_Z - \text{the function tan}) + \text{the function sec}$ is differentiable on Z , and
(ii) for every x such that $x \in Z$ holds $((\text{id}_Z - \text{the function tan}) + \text{the function sec})'_{\upharpoonright Z}(x) = \frac{\sin x}{\sin x + 1}$.
- (54) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{1 + \sin x}$,
(iii) $Z \subseteq \text{dom}((\text{id}_Z - \text{the function tan}) + \text{the function sec})$,
(iv) $Z = \text{dom } f$, and
(v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = ((\sup A - \tan \sup A) + \sec \sup A) - ((\inf A - \tan \inf A) + \sec \inf A)$.

(55) Suppose $Z \subseteq \text{dom}(\text{id}_Z - \text{the function tan} - \text{the function sec})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

- (i) $\text{id}_Z - \text{the function tan} - \text{the function sec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\text{id}_Z - \text{the function tan} - \text{the function sec})'_{|Z}(x) = \frac{\sin x}{\sin x - 1}$.

(56) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{\sin x - 1}$,
- (iii) $Z \subseteq \text{dom}(\text{id}_Z - \text{the function tan} - \text{the function sec})$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = \sup A - \tan \sup A - \sec \sup A - (\inf A - \tan \inf A - \sec \inf A)$.

(57) Suppose $Z \subseteq \text{dom}(\text{the function tan} - \text{id}_Z)$. Then $(\text{the function tan} - \text{id}_Z)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\text{the function tan} - \text{id}_Z)'_{|Z}(x) = (\tan x)^2$.

(58) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $(\text{the function cos})(x) > 0$ and $f(x) = (\tan x)^2$,
- (iii) $Z \subseteq \text{dom}(\text{the function tan} - \text{id}_Z)$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = \tan \sup A - \sup A - (\tan \inf A - \inf A)$.

(59) Suppose $Z \subseteq \text{dom}(-\text{the function cot} - \text{id}_Z)$. Then $-\text{the function cot} - \text{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\text{the function cot} - \text{id}_Z)'_{|Z}(x) = (\cot x)^2$.

(60) Suppose that

- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $(\text{the function sin})(x) > 0$ and $f(x) = (\cot x)^2$,
- (iii) $Z \subseteq \text{dom}(-\text{the function cot} - \text{id}_Z)$,
- (iv) $Z = \text{dom } f$, and
- (v) $f \upharpoonright A$ is continuous.

Then $\int_A f(x)dx = -\cot \sup A - \sup A - (-\cot \inf A - \inf A)$.

(61) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\cos x)^2}$ and $\cos x \neq 0$ and $\text{dom}(\text{the function } \tan) = Z = \text{dom } f$ and $f|_A$ is continuous.

Then $\int_A f(x)dx = \tan \sup A - \tan \inf A$.

(62) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\sin x)^2}$ and $\sin x \neq 0$ and $\text{dom}(\text{the function } \cot) = Z = \text{dom } f$ and $f|_A$ is continuous.

Then $\int_A f(x)dx = \cot \sup A - \cot \inf A$.

(63) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x - (\cos x)^2}{(\cos x)^2}$ and $Z \subseteq \text{dom}(\text{the function } \sec) - \text{id}_Z$ and $Z = \text{dom } f$ and $f|_A$ is continuous.

Then $\int_A f(x)dx = \sec \sup A - \sup A - (\sec \inf A - \inf A)$.

(64) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\cos x - (\sin x)^2}{(\sin x)^2}$,

(iii) $Z \subseteq \text{dom}(-\text{the function } \text{cosec} - \text{id}_Z)$,

(iv) $Z = \text{dom } f$, and

(v) $f|_A$ is continuous.

Then $\int_A f(x)dx = -\text{cosec} \sup A - \sup A - (-\text{cosec} \inf A - \inf A)$.

The following propositions are true:

(65) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $\sin x > 0$,

(iii) $Z \subseteq \text{dom}(\text{the function } \ln) \cdot \text{the function } \sin$,

(iv) $Z = \text{dom}(\text{the function } \cot)$, and

(v) $(\text{the function } \cot)|_A$ is continuous.

Then $\int_A (\text{the function } \cot)(x)dx = \ln \sin \sup A - \ln \sin \inf A$.

(66) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}$,

(iv) $Z \subseteq \text{dom}(\frac{1}{2}(\text{the function } \arcsin)^2)$,

(v) $Z = \text{dom } f$, and

(vi) $f|_A$ is continuous.

$$\text{Then } \int_A f(x)dx = \frac{1}{2} \cdot (\arcsin \sup A)^2 - \frac{1}{2} \cdot (\arcsin \inf A)^2.$$

(67) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = -\frac{\arccos x}{\sqrt{1-x^2}}$,
- (iv) $Z \subseteq \text{dom}(\frac{1}{2}(\text{the function arccos})^2)$,
- (v) $Z = \text{dom } f$, and
- (vi) $f \upharpoonright A$ is continuous.

$$\text{Then } \int_A f(x)dx = \frac{1}{2} \cdot (\arccos \sup A)^2 - \frac{1}{2} \cdot (\arccos \inf A)^2.$$

(68) $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and $\text{dom}(\text{the function arcsin}) = Z \subseteq \text{dom}(\text{id}_Z(\text{the function arcsin}) + f^{\frac{1}{2}})$.

(69) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^2$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and $\text{dom}((\text{the function arcsin}) \cdot f_3) = Z \subseteq \text{dom}(\text{id}_Z((\text{the function arcsin}) \cdot f_3) + (\square^{\frac{1}{2}}) \cdot f)$ and $((\text{the function arcsin}) \cdot f_3) \upharpoonright A$ is continuous. Then $\int_A ((\text{the function arcsin}) \cdot f_3)(x)dx =$

$$(\sup A \cdot \arcsin(\frac{\sup A}{a}) + f(\sup A)^{\frac{1}{2}}) - (\inf A \cdot \arcsin(\frac{\inf A}{a}) + f(\inf A)^{\frac{1}{2}}).$$

(70) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and $\text{dom}(\text{the function arccos}) = Z \subseteq \text{dom}(\text{id}_Z(\text{the function arccos}) - (\square^{\frac{1}{2}}) \cdot f)$.

$$\text{Then } \int_A (\text{the function arccos})(x)dx = \sup A \cdot \arccos \sup A - f(\sup A)^{\frac{1}{2}} - (\inf A \cdot \arccos \inf A - f(\inf A)^{\frac{1}{2}}).$$

(71) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^2$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and $\text{dom}((\text{the function arccos}) \cdot f_3) = Z = \text{dom}(\text{id}_Z((\text{the function arccos}) \cdot f_3) - (\square^{\frac{1}{2}}) \cdot f)$ and $((\text{the function arccos}) \cdot f_3) \upharpoonright A$ is continuous. Then $\int_A ((\text{the function arccos}) \cdot f_3)(x)dx =$

$$\sup A \cdot \arccos(\frac{\sup A}{a}) - f(\sup A)^{\frac{1}{2}} - (\inf A \cdot \arccos(\frac{\inf A}{a}) - f(\inf A)^{\frac{1}{2}}).$$

(72) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) $f_2 = \square^2$,

- (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
- (v) $Z = \text{dom}(\text{the function } \arctan)$, and
- (vi) $Z = \text{dom}(\text{id}_Z \text{ the function } \arctan - \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$.

$$\text{Then } \int_A (\text{the function } \arctan)(x) dx = \sup A \cdot \arctan \sup A - \frac{1}{2} \cdot \ln(1 + (\sup A)^2) - (\inf A \cdot \arctan \inf A - \frac{1}{2} \cdot \ln(1 + (\inf A)^2)).$$

(73) Suppose that

- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) $f_2 = \square^2$,
- (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
- (v) $\text{dom}(\text{the function } \text{arccot}) = Z$, and
- (vi) $Z = \text{dom}(\text{id}_Z \text{ the function } \text{arccot} + \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$.

$$\text{Then } \int_A (\text{the function } \text{arccot})(x) dx = (\sup A \cdot \text{arccot } \sup A + \frac{1}{2} \cdot \ln(1 + (\sup A)^2)) - (\inf A \cdot \text{arccot } \inf A + \frac{1}{2} \cdot \ln(1 + (\inf A)^2)).$$

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