

Arithmetic Operations on Functions from Sets into Functional Sets

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Summary. In this paper we introduce sets containing number-valued functions. Different arithmetic operations on maps between any set and such functional sets are later defined.

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The notation and terminology used here are introduced in the following papers: [4], [9], [10], [2], [11], [6], [3], [1], [8], [5], and [7].

1. FUNCTIONAL SETS

In this paper x , X , X_1 , X_2 are sets.

Let Y be a functional set. The functor $\text{DOMS}(Y)$ is defined by:

(Def. 1) $\text{DOMS}(Y) = \bigcup \{\text{dom } f : f \text{ ranges over elements of } Y\}$.

Let us consider X . We say that X is complex-functions-membered if and only if:

(Def. 2) If $x \in X$, then x is a complex-valued function.

Let us consider X . We say that X is extended-real-functions-membered if and only if:

(Def. 3) If $x \in X$, then x is an extended real-valued function.

Let us consider X . We say that X is real-functions-membered if and only if:

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(Def. 4) If $x \in X$, then x is a real-valued function.

Let us consider X . We say that X is rational-functions-membered if and only if:

(Def. 5) If $x \in X$, then x is a rational-valued function.

Let us consider X . We say that X is integer-functions-membered if and only if:

(Def. 6) If $x \in X$, then x is an integer-valued function.

Let us consider X . We say that X is natural-functions-membered if and only if:

(Def. 7) If $x \in X$, then x is a natural-valued function.

One can check the following observations:

- * every set which is natural-functions-membered is also integer-functions-membered,
- * every set which is integer-functions-membered is also rational-functions-membered,
- * every set which is rational-functions-membered is also real-functions-membered,
- * every set which is real-functions-membered is also complex-functions-membered, and
- * every set which is real-functions-membered is also extended-real-functions-membered.

Let us mention that every set which is empty is also natural-functions-membered.

Let f be a complex-valued function. Observe that $\{f\}$ is complex-functions-membered.

One can verify that every set which is complex-functions-membered is also functional and every set which is extended-real-functions-membered is also functional.

One can verify that there exists a set which is natural-functions-membered and non empty.

Let X be a complex-functions-membered set. One can verify that every subset of X is complex-functions-membered.

Let X be an extended-real-functions-membered set. Note that every subset of X is extended-real-functions-membered.

Let X be a real-functions-membered set. Note that every subset of X is real-functions-membered.

Let X be a rational-functions-membered set. Observe that every subset of X is rational-functions-membered.

Let X be an integer-functions-membered set. Note that every subset of X is integer-functions-membered.

Let X be a natural-functions-membered set. Observe that every subset of X is natural-functions-membered.

Let D be a set. The functor $\mathbb{C}\text{-PFunCs } D$ yields a set and is defined by:

(Def. 8) For every set f holds $f \in \mathbb{C}\text{-PFunCs } D$ iff f is a partial function from D to \mathbb{C} .

Let D be a set. The functor $\mathbb{C}\text{-FunCs } D$ yielding a set is defined by:

(Def. 9) For every set f holds $f \in \mathbb{C}\text{-FunCs } D$ iff f is a function from D into \mathbb{C} .

Let D be a set. The functor $\overline{\mathbb{R}}\text{-PFunCs } D$ yields a set and is defined by:

(Def. 10) For every set f holds $f \in \overline{\mathbb{R}}\text{-PFunCs } D$ iff f is a partial function from D to $\overline{\mathbb{R}}$.

Let D be a set. The functor $\overline{\mathbb{R}}\text{-FunCs } D$ yields a set and is defined as follows:

(Def. 11) For every set f holds $f \in \overline{\mathbb{R}}\text{-FunCs } D$ iff f is a function from D into $\overline{\mathbb{R}}$.

Let D be a set. The functor $\mathbb{R}\text{-PFunCs } D$ yielding a set is defined by:

(Def. 12) For every set f holds $f \in \mathbb{R}\text{-PFunCs } D$ iff f is a partial function from D to \mathbb{R} .

Let D be a set. The functor $\mathbb{R}\text{-FunCs } D$ yielding a set is defined by:

(Def. 13) For every set f holds $f \in \mathbb{R}\text{-FunCs } D$ iff f is a function from D into \mathbb{R} .

Let D be a set. The functor $\mathbb{Q}\text{-PFunCs } D$ yields a set and is defined as follows:

(Def. 14) For every set f holds $f \in \mathbb{Q}\text{-PFunCs } D$ iff f is a partial function from D to \mathbb{Q} .

Let D be a set. The functor $\mathbb{Q}\text{-FunCs } D$ yields a set and is defined by:

(Def. 15) For every set f holds $f \in \mathbb{Q}\text{-FunCs } D$ iff f is a function from D into \mathbb{Q} .

Let D be a set. The functor $\mathbb{Z}\text{-PFunCs } D$ yielding a set is defined by:

(Def. 16) For every set f holds $f \in \mathbb{Z}\text{-PFunCs } D$ iff f is a partial function from D to \mathbb{Z} .

Let D be a set. The functor $\mathbb{Z}\text{-FunCs } D$ yields a set and is defined as follows:

(Def. 17) For every set f holds $f \in \mathbb{Z}\text{-FunCs } D$ iff f is a function from D into \mathbb{Z} .

Let D be a set. The functor $\mathbb{N}\text{-PFunCs } D$ yields a set and is defined by:

(Def. 18) For every set f holds $f \in \mathbb{N}\text{-PFunCs } D$ iff f is a partial function from D to \mathbb{N} .

Let D be a set. The functor $\mathbb{N}\text{-FunCs } D$ yielding a set is defined by:

(Def. 19) For every set f holds $f \in \mathbb{N}\text{-FunCs } D$ iff f is a function from D into \mathbb{N} .

The following propositions are true:

- (1) $\mathbb{C}\text{-FunCs } X$ is a subset of $\mathbb{C}\text{-PFunCs } X$.
- (2) $\overline{\mathbb{R}}\text{-FunCs } X$ is a subset of $\overline{\mathbb{R}}\text{-PFunCs } X$.
- (3) $\mathbb{R}\text{-FunCs } X$ is a subset of $\mathbb{R}\text{-PFunCs } X$.
- (4) $\mathbb{Q}\text{-FunCs } X$ is a subset of $\mathbb{Q}\text{-PFunCs } X$.
- (5) $\mathbb{Z}\text{-FunCs } X$ is a subset of $\mathbb{Z}\text{-PFunCs } X$.

(6) \mathbb{N} -Funcs X is a subset of \mathbb{N} -PFuncs X .

Let us consider X . One can verify the following observations:

- * \mathbb{C} -PFuncs X is complex-functions-membered,
- * \mathbb{C} -Funcs X is complex-functions-membered,
- * $\overline{\mathbb{R}}$ -PFuncs X is extended-real-functions-membered,
- * $\overline{\mathbb{R}}$ -Funcs X is extended-real-functions-membered,
- * \mathbb{R} -PFuncs X is real-functions-membered,
- * \mathbb{R} -Funcs X is real-functions-membered,
- * \mathbb{Q} -PFuncs X is rational-functions-membered,
- * \mathbb{Q} -Funcs X is rational-functions-membered,
- * \mathbb{Z} -PFuncs X is integer-functions-membered,
- * \mathbb{Z} -Funcs X is integer-functions-membered,
- * \mathbb{N} -PFuncs X is natural-functions-membered, and
- * \mathbb{N} -Funcs X is natural-functions-membered.

Let X be a complex-functions-membered set. Observe that every element of X is complex-valued.

Let X be an extended-real-functions-membered set. One can check that every element of X is extended real-valued.

Let X be a real-functions-membered set. One can check that every element of X is real-valued.

Let X be a rational-functions-membered set. One can check that every element of X is rational-valued.

Let X be an integer-functions-membered set. Observe that every element of X is integer-valued.

Let X be a natural-functions-membered set. Observe that every element of X is natural-valued.

Let X, x be sets, let Y be a complex-functions-membered set, and let f be a partial function from X to Y . Observe that $f(x)$ is function-like and relation-like.

Let X, x be sets, let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y . Observe that $f(x)$ is function-like and relation-like.

Let us consider X, x , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . One can check that $f(x)$ is complex-valued.

Let us consider X, x , let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y . One can verify that $f(x)$ is extended real-valued.

Let us consider X, x , let Y be a real-functions-membered set, and let f be a partial function from X to Y . Note that $f(x)$ is real-valued.

Let us consider X , x , let Y be a rational-functions-membered set, and let f be a partial function from X to Y . Note that $f(x)$ is rational-valued.

Let us consider X , x , let Y be an integer-functions-membered set, and let f be a partial function from X to Y . Note that $f(x)$ is integer-valued.

Let us consider X , x , let Y be a natural-functions-membered set, and let f be a partial function from X to Y . One can check that $f(x)$ is natural-valued.

Let us consider X and let Y be a complex-membered set. One can check that $X \dot{\rightarrow} Y$ is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Observe that $X \dot{\rightarrow} Y$ is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Observe that $X \dot{\rightarrow} Y$ is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Observe that $X \dot{\rightarrow} Y$ is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Observe that $X \dot{\rightarrow} Y$ is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can verify that $X \dot{\rightarrow} Y$ is natural-functions-membered.

Let us consider X and let Y be a complex-membered set. Note that Y^X is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Note that Y^X is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Note that Y^X is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Note that Y^X is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Note that Y^X is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can check that Y^X is natural-functions-membered.

Let R be a binary relation. We say that R is complex-functions-valued if and only if:

(Def. 20) $\text{rng } R$ is complex-functions-membered.

We say that R is extended-real-functions-valued if and only if:

(Def. 21) $\text{rng } R$ is extended-real-functions-membered.

We say that R is real-functions-valued if and only if:

(Def. 22) $\text{rng } R$ is real-functions-membered.

We say that R is rational-functions-valued if and only if:

(Def. 23) $\text{rng } R$ is rational-functions-membered.

We say that R is integer-functions-valued if and only if:

(Def. 24) $\text{rng } R$ is integer-functions-membered.

We say that R is natural-functions-valued if and only if:

(Def. 25) $\text{rng } R$ is natural-functions-membered.

Let f be a function. Let us observe that f is complex-functions-valued if and only if:

(Def. 26) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is a complex-valued function.

Let us observe that f is extended-real-functions-valued if and only if:

(Def. 27) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is an extended real-valued function.

Let us observe that f is real-functions-valued if and only if:

(Def. 28) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is a real-valued function.

Let us observe that f is rational-functions-valued if and only if:

(Def. 29) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is a rational-valued function.

Let us observe that f is integer-functions-valued if and only if:

(Def. 30) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is an integer-valued function.

Let us observe that f is natural-functions-valued if and only if:

(Def. 31) For every set x such that $x \in \text{dom } f$ holds $f(x)$ is a natural-valued function.

One can verify the following observations:

- * every binary relation which is natural-functions-valued is also integer-functions-valued,
- * every binary relation which is integer-functions-valued is also rational-functions-valued,
- * every binary relation which is rational-functions-valued is also real-functions-valued,
- * every binary relation which is real-functions-valued is also extended-real-functions-valued, and
- * every binary relation which is real-functions-valued is also complex-functions-valued.

Let us note that every binary relation which is empty is also natural-functions-valued.

Let us mention that there exists a function which is natural-functions-valued.

Let R be a complex-functions-valued binary relation. Note that $\text{rng } R$ is complex-functions-membered.

Let R be an extended-real-functions-valued binary relation. Observe that $\text{rng } R$ is extended-real-functions-membered.

Let R be a real-functions-valued binary relation. Note that $\text{rng } R$ is real-functions-membered.

Let R be a rational-functions-valued binary relation. Observe that $\text{rng } R$ is rational-functions-membered.

Let R be an integer-functions-valued binary relation. One can verify that $\text{rng } R$ is integer-functions-membered.

Let R be a natural-functions-valued binary relation. One can check that $\text{rng } R$ is natural-functions-membered.

Let us consider X and let Y be a complex-functions-membered set. Observe that every partial function from X to Y is complex-functions-valued.

Let us consider X and let Y be an extended-real-functions-membered set. One can check that every partial function from X to Y is extended-real-functions-valued.

Let us consider X and let Y be a real-functions-membered set. One can check that every partial function from X to Y is real-functions-valued.

Let us consider X and let Y be a rational-functions-membered set. Observe that every partial function from X to Y is rational-functions-valued.

Let us consider X and let Y be an integer-functions-membered set. Observe that every partial function from X to Y is integer-functions-valued.

Let us consider X and let Y be a natural-functions-membered set. Note that every partial function from X to Y is natural-functions-valued.

Let f be a complex-functions-valued function and let us consider x . Note that $f(x)$ is function-like and relation-like.

Let f be an extended-real-functions-valued function and let us consider x . Observe that $f(x)$ is function-like and relation-like.

Let f be a complex-functions-valued function and let us consider x . One can verify that $f(x)$ is complex-valued.

Let f be an extended-real-functions-valued function and let us consider x . Note that $f(x)$ is extended real-valued.

Let f be a real-functions-valued function and let us consider x . One can verify that $f(x)$ is real-valued.

Let f be a rational-functions-valued function and let us consider x . Observe that $f(x)$ is rational-valued.

Let f be an integer-functions-valued function and let us consider x . Note that $f(x)$ is integer-valued.

Let f be a natural-functions-valued function and let us consider x . One can check that $f(x)$ is natural-valued.

2. OPERATIONS

For simplicity, we adopt the following rules: Y, Y_1, Y_2 are complex-functions-membered sets, c, c_1, c_2 are complex numbers, f is a partial function from X

to Y , f_1 is a partial function from X_1 to Y_1 , f_2 is a partial function from X_2 to Y_2 , and g, h, k are complex-valued functions.

We now state a number of propositions:

- (7) If $g \neq \emptyset$ and $g + c_1 = g + c_2$, then $c_1 = c_2$.
- (8) If $g \neq \emptyset$ and $g - c_1 = g - c_2$, then $c_1 = c_2$.
- (9) If $g \neq \emptyset$ and g is non-empty and $g c_1 = g c_2$, then $c_1 = c_2$.
- (10) $-(g + c) = -g - c$.
- (11) $-(g - c) = -g + c$.
- (12) $(g + c_1) + c_2 = g + (c_1 + c_2)$.
- (13) $(g + c_1) - c_2 = g + (c_1 - c_2)$.
- (14) $(g - c_1) + c_2 = g - (c_1 - c_2)$.
- (15) $g - c_1 - c_2 = g - (c_1 + c_2)$.
- (16) $g c_1 c_2 = g (c_1 \cdot c_2)$.
- (17) $-(g + h) = -g - h$.
- (18) $g - h = -(h - g)$.
- (19) $(g h)/k = g (h/k)$.
- (20) $(g/h) k = (g k)/h$.
- (21) $g/h/k = g/(h k)$.
- (22) $c - g = (-c) g$.
- (23) $c - g = -c g$.
- (24) $(-c) g = -c g$.
- (25) $-g h = (-g) h$.
- (26) $-g/h = (-g)/h$.
- (27) $-g/h = g/-h$.

Let f be a complex-valued function and let c be a complex number. The functor f/c yields a function and is defined as follows:

(Def. 32) $f/c = \frac{1}{c} f$.

Let f be a complex-valued function and let c be a complex number. Note that f/c is complex-valued.

Let f be a real-valued function and let r be a real number. Note that f/r is real-valued.

Let f be a rational-valued function and let r be a rational number. One can check that f/r is rational-valued.

Let f be a complex-valued finite sequence and let c be a complex number. One can check that f/c is finite sequence-like.

The following propositions are true:

- (28) $\text{dom}(g/c) = \text{dom } g$.
- (29) $(g/c)(x) = \frac{g(x)}{c}$.

- (30) $(-g)/c = -g/c$.
 (31) $g/-c = -g/c$.
 (32) $g/-c = (-g)/c$.
 (33) If $g \neq \emptyset$ and g is non-empty and $g/c_1 = g/c_2$, then $c_1 = c_2$.
 (34) $(g c_1)/c_2 = g \frac{c_1}{c_2}$.
 (35) $(g/c_1) c_2 = (g c_2)/c_1$.
 (36) $g/c_1/c_2 = g/(c_1 \cdot c_2)$.
 (37) $(g + h)/c = g/c + h/c$.
 (38) $(g - h)/c = g/c - h/c$.
 (39) $(g h)/c = g(h/c)$.
 (40) $(g/h)/c = g/(h c)$.

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . The functor $-f$ yields a function and is defined by:

- (Def. 33) $\text{dom}(-f) = \text{dom } f$ and for every set x such that $x \in \text{dom}(-f)$ holds $(-f)(x) = -f(x)$.

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . Then $-f$ is a partial function from X to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, and let f be a partial function from X to Y . Then $-f$ is a partial function from X to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, and let f be a partial function from X to Y . Then $-f$ is a partial function from X to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be an integer-functions-membered set, and let f be a partial function from X to Y . Then $-f$ is a partial function from X to $\mathbb{Z}\text{-PFunCS DOMS}(Y)$.

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y . One can check that $-f$ is finite sequence-like.

We now state two propositions:

- (41) $--f = f$.
 (42) If $-f_1 = -f_2$, then $f_1 = f_2$.

Let X be a complex-functions-membered set, let Y be a set, and let f be a partial function from X to Y . The functor $f \circ -$ yielding a function is defined as follows:

- (Def. 34) $\text{dom}(f \circ -) = \text{dom } f$ and for every complex-valued function x such that $x \in \text{dom}(f \circ -)$ holds $(f \circ -)(x) = f(-x)$.

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . The functor ${}^1/f$ yields a function and is defined as follows:

(Def. 35) $\text{dom } {}^1/f = \text{dom } f$ and for every set x such that $x \in \text{dom } {}^1/f$ holds $({}^1/f)(x) = f(x)^{-1}$.

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . Then ${}^1/f$ is a partial function from X to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, and let f be a partial function from X to Y . Then ${}^1/f$ is a partial function from X to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, and let f be a partial function from X to Y . Then ${}^1/f$ is a partial function from X to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y . Note that ${}^1/f$ is finite sequence-like.

The following proposition is true

$$(43) \quad {}^1/{}^1/f = f.$$

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . The functor $|f|$ yields a function and is defined by:

(Def. 36) $\text{dom } |f| = \text{dom } f$ and for every set x such that $x \in \text{dom } |f|$ holds $|f|(x) = |f(x)|$.

Let us consider X , let Y be a complex-functions-membered set, and let f be a partial function from X to Y . Then $|f|$ is a partial function from X to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, and let f be a partial function from X to Y . Then $|f|$ is a partial function from X to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, and let f be a partial function from X to Y . Then $|f|$ is a partial function from X to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be an integer-functions-membered set, and let f be a partial function from X to Y . Then $|f|$ is a partial function from X to $\mathbb{N}\text{-PFunCS DOMS}(Y)$.

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y . Note that $|f|$ is finite sequence-like.

We now state the proposition

$$(44) \quad ||f|| = |f|.$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. The functor $f + c$

yields a function and is defined by:

(Def. 37) $\text{dom}(f + c) = \text{dom } f$ and for every set x such that $x \in \text{dom}(f + c)$ holds
 $(f + c)(x) = c + f(x)$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. Then $f + c$ is a partial function from X to \mathbb{C} -PFunks DOMS(Y).

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let c be a real number. Then $f + c$ is a partial function from X to \mathbb{R} -PFunks DOMS(Y).

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let c be a rational number. Then $f + c$ is a partial function from X to \mathbb{Q} -PFunks DOMS(Y).

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let c be an integer number. Then $f + c$ is a partial function from X to \mathbb{Z} -PFunks DOMS(Y).

Let us consider X , let Y be a natural-functions-membered set, let f be a partial function from X to Y , and let c be a natural number. Then $f + c$ is a partial function from X to \mathbb{N} -PFunks DOMS(Y).

One can prove the following propositions:

$$(45) \quad f + c_1 + c_2 = f + (c_1 + c_2).$$

$$(46) \quad \text{If } f \neq \emptyset \text{ and } f \text{ is non-empty and } f + c_1 = f + c_2, \text{ then } c_1 = c_2.$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. The functor $f - c$ yields a function and is defined as follows:

(Def. 38) $f - c = f + -c$.

We now state two propositions:

$$(47) \quad \text{dom}(f - c) = \text{dom } f.$$

$$(48) \quad \text{If } x \in \text{dom}(f - c), \text{ then } (f - c)(x) = f(x) - c.$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. Then $f - c$ is a partial function from X to \mathbb{C} -PFunks DOMS(Y).

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let c be a real number. Then $f - c$ is a partial function from X to \mathbb{R} -PFunks DOMS(Y).

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let c be a rational number. Then $f - c$ is a partial function from X to \mathbb{Q} -PFunks DOMS(Y).

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let c be an integer number. Then $f - c$ is a partial function from X to \mathbb{Z} -PFunks DOMS(Y).

We now state four propositions:

(49) If $f \neq \emptyset$ and f is non-empty and $f - c_1 = f - c_2$, then $c_1 = c_2$.

(50) $(f + c_1) - c_2 = f + (c_1 - c_2)$.

(51) $(f - c_1) + c_2 = f - (c_1 - c_2)$.

(52) $f - c_1 - c_2 = f - (c_1 + c_2)$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. The functor $f \cdot c$ yielding a function is defined as follows:

(Def. 39) $\text{dom}(f \cdot c) = \text{dom } f$ and for every set x such that $x \in \text{dom}(f \cdot c)$ holds $(f \cdot c)(x) = c f(x)$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. Then $f \cdot c$ is a partial function from X to $\mathbb{C}\text{-PFuncs DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let c be a real number. Then $f \cdot c$ is a partial function from X to $\mathbb{R}\text{-PFuncs DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let c be a rational number. Then $f \cdot c$ is a partial function from X to $\mathbb{Q}\text{-PFuncs DOMS}(Y)$.

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let c be an integer number. Then $f \cdot c$ is a partial function from X to $\mathbb{Z}\text{-PFuncs DOMS}(Y)$.

Let us consider X , let Y be a natural-functions-membered set, let f be a partial function from X to Y , and let c be a natural number. Then $f \cdot c$ is a partial function from X to $\mathbb{N}\text{-PFuncs DOMS}(Y)$.

The following two propositions are true:

(53) $f \cdot c_1 \cdot c_2 = f \cdot (c_1 \cdot c_2)$.

(54) If $f \neq \emptyset$ and f is non-empty and for every x such that $x \in \text{dom } f$ holds $f(x)$ is non-empty and $f \cdot c_1 = f \cdot c_2$, then $c_1 = c_2$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. The functor f/c yielding a function is defined as follows:

(Def. 40) $f/c = f \cdot c^{-1}$.

One can prove the following propositions:

(55) $\text{dom}(f/c) = \text{dom } f$.

(56) If $x \in \text{dom}(f/c)$, then $(f/c)(x) = c^{-1} f(x)$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let c be a complex number. Then f/c is a partial function from X to $\mathbb{C}\text{-PFuncs DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let c be a real number. Then f/c is a partial function from X to \mathbb{R} -PFuncs DOMS(Y).

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let c be a rational number. Then f/c is a partial function from X to \mathbb{Q} -PFuncs DOMS(Y).

The following propositions are true:

$$(57) \quad f/c_1/c_2 = f/(c_1 \cdot c_2).$$

$$(58) \quad \text{If } f \neq \emptyset \text{ and } f \text{ is non-empty and for every } x \text{ such that } x \in \text{dom } f \text{ holds } f(x) \text{ is non-empty and } f/c_1 = f/c_2, \text{ then } c_1 = c_2.$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. The functor $f + g$ yielding a function is defined as follows:

$$(\text{Def. 41}) \quad \text{dom}(f+g) = \text{dom } f \cap \text{dom } g \text{ and for every set } x \text{ such that } x \in \text{dom}(f+g) \text{ holds } (f+g)(x) = f(x) + g(x).$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. Then $f + g$ is a partial function from $X \cap \text{dom } g$ to \mathbb{C} -PFuncs DOMS(Y).

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let g be a real-valued function. Then $f + g$ is a partial function from $X \cap \text{dom } g$ to \mathbb{R} -PFuncs DOMS(Y).

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let g be a rational-valued function. Then $f + g$ is a partial function from $X \cap \text{dom } g$ to \mathbb{Q} -PFuncs DOMS(Y).

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let g be an integer-valued function. Then $f + g$ is a partial function from $X \cap \text{dom } g$ to \mathbb{Z} -PFuncs DOMS(Y).

Let us consider X , let Y be a natural-functions-membered set, let f be a partial function from X to Y , and let g be a natural-valued function. Then $f + g$ is a partial function from $X \cap \text{dom } g$ to \mathbb{N} -PFuncs DOMS(Y).

Next we state two propositions:

$$(59) \quad f + g + h = f + (g + h).$$

$$(60) \quad -(f + g) = (-f) + -g.$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. The functor $f - g$ yields a function and is defined by:

$$(\text{Def. 42}) \quad f - g = f + -g.$$

We now state two propositions:

$$(61) \quad \text{dom}(f - g) = \text{dom } f \cap \text{dom } g.$$

$$(62) \quad \text{If } x \in \text{dom}(f - g), \text{ then } (f - g)(x) = f(x) - g(x).$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. Then $f - g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let g be a real-valued function. Then $f - g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let g be a rational-valued function. Then $f - g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let g be an integer-valued function. Then $f - g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{Z}\text{-PFunCS DOMS}(Y)$.

The following propositions are true:

$$(63) \quad f - -g = f + g.$$

$$(64) \quad -(f - g) = (-f) + g.$$

$$(65) \quad (f + g) - h = f + (g - h).$$

$$(66) \quad (f - g) + h = f - (g - h).$$

$$(67) \quad f - g - h = f - (g + h).$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. The functor $f \cdot g$ yielding a function is defined by:

(Def. 43) $\text{dom}(f \cdot g) = \text{dom } f \cap \text{dom } g$ and for every set x such that $x \in \text{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x) g(x)$.

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. Then $f \cdot g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let g be a real-valued function. Then $f \cdot g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let g be a rational-valued function. Then $f \cdot g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be an integer-functions-membered set, let f be a partial function from X to Y , and let g be an integer-valued function. Then $f \cdot g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{Z}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a natural-functions-membered set, let f be a partial function from X to Y , and let g be a natural-valued function. Then $f \cdot g$ is a partial function from $X \cap \text{dom } g$ to $\mathbb{N}\text{-PFunCS DOMS}(Y)$.

Next we state three propositions:

$$(68) \quad f \cdot -g = (-f) \cdot g.$$

$$(69) \quad f \cdot -g = -f \cdot g.$$

$$(70) \quad f \cdot g \cdot h = f \cdot (gh).$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. The functor f/g yields a function and is defined by:

$$(\text{Def. 44}) \quad f/g = f \cdot g^{-1}.$$

Next we state two propositions:

$$(71) \quad \text{dom}(f/g) = \text{dom } f \cap \text{dom } g.$$

$$(72) \quad \text{If } x \in \text{dom}(f/g), \text{ then } (f/g)(x) = f(x)/g(x).$$

Let us consider X , let Y be a complex-functions-membered set, let f be a partial function from X to Y , and let g be a complex-valued function. Then f/g is a partial function from $X \cap \text{dom } g$ to $\mathbb{C}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a real-functions-membered set, let f be a partial function from X to Y , and let g be a real-valued function. Then f/g is a partial function from $X \cap \text{dom } g$ to $\mathbb{R}\text{-PFunCS DOMS}(Y)$.

Let us consider X , let Y be a rational-functions-membered set, let f be a partial function from X to Y , and let g be a rational-valued function. Then f/g is a partial function from $X \cap \text{dom } g$ to $\mathbb{Q}\text{-PFunCS DOMS}(Y)$.

Next we state the proposition

$$(73) \quad (f \cdot g)/h = f \cdot (g/h).$$

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . The functor $f + g$ yielding a function is defined as follows:

$$(\text{Def. 45}) \quad \text{dom}(f+g) = \text{dom } f \cap \text{dom } g \text{ and for every set } x \text{ such that } x \in \text{dom}(f+g) \\ \text{holds } (f+g)(x) = f(x) + g(x).$$

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}\text{-PFunCS}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}\text{-PFunCS}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}\text{-PFunCS}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to

Y_2 . Then $f + g$ is a partial function from $X_1 \cap X_2$ to \mathbb{Z} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f + g$ is a partial function from $X_1 \cap X_2$ to \mathbb{N} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

We now state three propositions:

$$(74) \quad f_1 + f_2 = f_2 + f_1.$$

$$(75) \quad (f + f_1) + f_2 = f + (f_1 + f_2).$$

$$(76) \quad -(f_1 + f_2) = (-f_1) + -f_2.$$

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . The functor $f - g$ yields a function and is defined by:

(Def. 46) $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$ and for every set x such that $x \in \text{dom}(f - g)$ holds $(f - g)(x) = f(x) - g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f - g$ is a partial function from $X_1 \cap X_2$ to \mathbb{C} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f - g$ is a partial function from $X_1 \cap X_2$ to \mathbb{R} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f - g$ is a partial function from $X_1 \cap X_2$ to \mathbb{Q} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f - g$ is a partial function from $X_1 \cap X_2$ to \mathbb{Z} -PFuncs(DOMS(Y_1) \cap DOMS(Y_2)).

One can prove the following propositions:

$$(77) \quad f_1 - f_2 = -(f_2 - f_1).$$

$$(78) \quad -(f_1 - f_2) = (-f_1) + f_2.$$

$$(79) \quad (f + f_1) - f_2 = f + (f_1 - f_2).$$

$$(80) \quad (f - f_1) + f_2 = f - (f_1 - f_2).$$

$$(81) \quad f - f_1 - f_2 = f - (f_1 + f_2).$$

$$(82) \quad f - f_1 - f_2 = f - f_2 - f_1.$$

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 .

The functor $f \cdot g$ yields a function and is defined by:

(Def. 47) $\text{dom}(f \cdot g) = \text{dom } f \cap \text{dom } g$ and for every set x such that $x \in \text{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x)g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{N}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

We now state several propositions:

$$(83) \quad f_1 \cdot f_2 = f_2 \cdot f_1.$$

$$(84) \quad (f \cdot f_1) \cdot f_2 = f \cdot (f_1 \cdot f_2).$$

$$(85) \quad (-f_1) \cdot f_2 = -f_1 \cdot f_2.$$

$$(86) \quad f_1 \cdot -f_2 = -f_1 \cdot f_2.$$

$$(87) \quad f \cdot (f_1 + f_2) = f \cdot f_1 + f \cdot f_2.$$

$$(88) \quad (f_1 + f_2) \cdot f = f_1 \cdot f + f_2 \cdot f.$$

$$(89) \quad f \cdot (f_1 - f_2) = f \cdot f_1 - f \cdot f_2.$$

$$(90) \quad (f_1 - f_2) \cdot f = f_1 \cdot f - f_2 \cdot f.$$

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . The functor f/g yields a function and is defined by:

(Def. 48) $\text{dom}(f/g) = \text{dom } f \cap \text{dom } g$ and for every set x such that $x \in \text{dom}(f/g)$ holds $(f/g)(x) = f(x)/g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2

to Y_2 . Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{C}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{R}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1 , and let g be a partial function from X_2 to Y_2 . Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}\text{-PFuncs}(\text{DOMS}(Y_1) \cap \text{DOMS}(Y_2))$.

One can prove the following propositions:

- (91) $(-f_1)/f_2 = -f_1/f_2$.
- (92) $f_1/-f_2 = -f_1/f_2$.
- (93) $(f \cdot f_1)/f_2 = f \cdot (f_1/f_2)$.
- (94) $(f/f_1) \cdot f_2 = (f \cdot f_2)/f_1$.
- (95) $f/f_1/f_2 = f/(f_1 \cdot f_2)$.
- (96) $(f_1 + f_2)/f = f_1/f + f_2/f$.
- (97) $(f_1 - f_2)/f = f_1/f - f_2/f$.

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