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Some Operations on Quaternion Numbers

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Summary. In this article, we give some equality and basic theorems about quaternion numbers, and some special operations.

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The articles [11], [1], [12], [3], [4], [9], [2], [5], [8], [7], [10], [13], and [6] provide the notation and terminology for this paper.

In this paper z_1, z_2, z_3, z_4, z are quaternion numbers.

The following propositions are true:

(1) $\Re(z_1 \cdot z_2) = \Re(z_2 \cdot z_1)$.

(2) If z is a real number, then $z + z_3 = \Re(z) + \Re(z_3) + \Im_1(z_3) \cdot i + \Im_2(z_3) \cdot j + \Im_3(z_3) \cdot k$.

(3) If z is a real number, then $z - z_3 = \langle \Re(z) - \Re(z_3), -\Im_1(z_3), -\Im_2(z_3), -\Im_3(z_3) \rangle_{\mathbb{H}}$.

(4) If z is a real number, then $z \cdot z_3 = \langle \Re(z) \cdot \Re(z_3), \Re(z) \cdot \Im_1(z_3), \Re(z) \cdot \Im_2(z_3), \Re(z) \cdot \Im_3(z_3) \rangle_{\mathbb{H}}$.

(5) If z is a real number, then $z \cdot i = \langle 0, \Re(z), 0, 0 \rangle_{\mathbb{H}}$.

(6) If z is a real number, then $z \cdot j = \langle 0, 0, \Re(z), 0 \rangle_{\mathbb{H}}$.

(7) If z is a real number, then $z \cdot k = \langle 0, 0, 0, \Re(z) \rangle_{\mathbb{H}}$.

(8) $z - 0_{\mathbb{H}} = z$.

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(9) If z is a real number, then $z \cdot z_1 = z_1 \cdot z$.
(10) If $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $z = \Re(z)$.
(11) $|z|^{\mathbf{2}} = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.
(12) $|z|^{\mathbf{2}} = |z \cdot \overline{z}|$.
(13) $|z|^{\mathbf{2}} = \Re(z \cdot \overline{z})$.
(14) $2 \cdot \Re(z) = \Re(z + \overline{z})$.
(15) If z is a real number, then $\overline{z \cdot z_1} = \overline{z} \cdot \overline{z_1}$.
(16) $\overline{z_1 \cdot z_2} = \overline{z_2} \cdot \overline{z_1}$.
(17) $|z_1 \cdot z_2|^{\mathbf{2}} = |z_1|^{\mathbf{2}} \cdot |z_2|^{\mathbf{2}}$.
(18) $i \cdot z_1 - z_1 \cdot i = \langle 0, 0, -2 \cdot \Im_3(z_1), 2 \cdot \Im_2(z_1)\rangle_{\mathbb{H}}$.
(19) $i \cdot z_1 + z_1 \cdot i = \langle -2 \cdot \Im_1(z_1), 2 \cdot \Re(z_1), 0, 0\rangle_{\mathbb{H}}$.
(20) $j \cdot z_1 - z_1 \cdot j = \langle 0, 2 \cdot \Im_3(z_1), 0, -2 \cdot \Im_1(z_1)\rangle_{\mathbb{H}}$.
(21) $j \cdot z_1 + z_1 \cdot j = \langle -2 \cdot \Im_2(z_1), 0, 2 \cdot \Re(z_1), 0\rangle_{\mathbb{H}}$.
(22) $k \cdot z_1 - z_1 \cdot k = \langle 0, -2 \cdot \Im_2(z_1), 2 \cdot \Im_1(z_1), 0\rangle_{\mathbb{H}}$.
(23) $k \cdot z_1 + z_1 \cdot k = \langle -2 \cdot \Im_3(z_1), 0, 0, 2 \cdot \Re(z_1)\rangle_{\mathbb{H}}$.
(24) $\Re(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = \frac{1}{|z|^{\mathbf{2}}} \cdot \Re(z)$.
(25) $\Im_1(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_1(z)$.
(26) $\Im_2(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_2(z)$.
(27) $\Im_3(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_3(z)$.
(28) $\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z} = \langle \frac{1}{|z|^{\mathbf{2}}} \cdot \Re(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_1(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_2(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_3(z)\rangle_{\mathbb{H}}$.
(29) $z \cdot (\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = \langle \frac{|z|^{\mathbf{2}}}{|z|^{\mathbf{2}}}, 0, 0, 0\rangle_{\mathbb{H}}$.
(30) $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$.
(31) $\Im_1(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)$.
(32) $\Im_2(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)$.
(33) $\Im_3(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)$.
(34) $|z_1 \cdot z_2 \cdot z_3|^{\mathbf{2}} = |z_1|^{\mathbf{2}} \cdot |z_2|^{\mathbf{2}} \cdot |z_3|^{\mathbf{2}}$.
(35) $\Re(z_1 \cdot z_2 \cdot z_3) = \Re(z_3 \cdot z_1 \cdot z_2)$.
(36) $|z \cdot z| = |\overline{z} \cdot \overline{z}|$.
(37) $|\overline{z} \cdot \overline{z}| = |z|^{\mathbf{2}}$.
(38) $|z_1 \cdot z_2 \cdot z_3| = |z_1| \cdot |z_2| \cdot |z_3|$.
(39) $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$.
(40) $|(z_1 + z_2) - z_3| \leq |z_1| + |z_2| + |z_3|$.
(41) $|z_1 - z_2 - z_3| \leq |z_1| + |z_2| + |z_3|$.

(42) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.

(43) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.

(44) $||z_1| - |z_2|| \leq |z_2 - z_1|$.

(45) $||z_1| - |z_2|| \leq |z_1| + |z_2|$.

(46) $|z_1| - |z_2| \leq |z_1 - z| + |z - z_2|$.

(47) If $|z_1| - |z_2| \neq 0$, then $(|z_1|^{\mathbf{2}} + |z_2|^{\mathbf{2}}) - 2 \cdot |z_1| \cdot |z_2| > 0$.

(48) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.

(49) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.

(50) $(z_1 \cdot z_2)^{-1} = {z_2}^{-1} \cdot {z_1}^{-1}$.

(51) $\overline{z}^{-1} = \overline{z^{-1}}$.

(52) $(1_{\mathbb{H}})^{-1} = 1_{\mathbb{H}}$.

(53) If $|z_1| = |z_2|$ and $|z_1| \neq 0$ and ${z_1}^{-1} = {z_2}^{-1}$, then $z_1 = z_2$.

(54) $(z_1 - z_2) \cdot (z_3 + z_4) = ((z_1 \cdot z_3 - z_2 \cdot z_3) + z_1 \cdot z_4) - z_2 \cdot z_4$.

(55) $(z_1 + z_2) \cdot (z_3 + z_4) = z_1 \cdot z_3 + z_2 \cdot z_3 + z_1 \cdot z_4 + z_2 \cdot z_4$.

(56) $-(z_1 + z_2) = -z_1 - z_2$.

(57) $-(z_1 - z_2) = -z_1 + z_2$.

(58) $z - (z_1 + z_2) = z - z_1 - z_2$.

(59) $z - (z_1 - z_2) = (z - z_1) + z_2$.

(60) $(z_1 + z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 + z_2 \cdot z_3) - z_1 \cdot z_4 - z_2 \cdot z_4$.

(61) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 - z_2 \cdot z_3 - z_1 \cdot z_4) + z_2 \cdot z_4$.

(62) $-(z_1 + z_2 + z_3) = -z_1 - z_2 - z_3$.

(63) $-(z_1 - z_2 - z_3) = -z_1 + z_2 + z_3$.

(64) $-((z_1 - z_2) + z_3) = (-z_1 + z_2) - z_3$.

(65) $-((z_1 + z_2) - z_3) = (-z_1 - z_2) + z_3$.

(66) If $z_1 + z = z_2 + z$, then $z_1 = z_2$.

(67) If $z_1 - z = z_2 - z$, then $z_1 = z_2$.

(68) $((z_1 + z_2) - z_3) \cdot z_4 = (z_1 \cdot z_4 + z_2 \cdot z_4) - z_3 \cdot z_4$.

(69) $((z_1 - z_2) + z_3) \cdot z_4 = (z_1 \cdot z_4 - z_2 \cdot z_4) + z_3 \cdot z_4$.

(70) $(z_1 - z_2 - z_3) \cdot z_4 = z_1 \cdot z_4 - z_2 \cdot z_4 - z_3 \cdot z_4$.

(71) $(z_1 + z_2 + z_3) \cdot z_4 = z_1 \cdot z_4 + z_2 \cdot z_4 + z_3 \cdot z_4$.

(72) $(z_1 - z_2) \cdot z_3 = (z_2 - z_1) \cdot -z_3$.

(73) $z_3 \cdot (z_1 - z_2) = (-z_3) \cdot (z_2 - z_1)$.

(74) $(z_1 - z_2 - z_3) + z_4 = (z_4 - z_3 - z_2) + z_1$.

(75) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_2 - z_1) \cdot (z_4 - z_3)$.

(76) $z - z_1 - z_2 = z - z_2 - z_1$.

(77) $z^{-1} = \langle \frac{\Re(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_1(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_2(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_3(z)}{|z|^{\mathbf{2}}} \rangle_{\mathbb{H}}$.

(78) $\frac{z_1}{z_2} = \langle \frac{\Re(z_2)\cdot\Re(z_1)+\Im_1(z_1)\cdot\Im_1(z_2)+\Im_2(z_2)\cdot\Im_2(z_1)+\Im_3(z_2)\cdot\Im_3(z_1)}{|z_2|^{\mathbf{2}}}, \frac{(\Re(z_2)\cdot\Im_1(z_1)-\Im_1(z_2)\cdot\Re(z_1)-\Im_2(z_2)\cdot\Im_3(z_1))+\Im_3(z_2)\cdot\Im_2(z_1)}{|z_2|^{\mathbf{2}}}, \frac{(\Re(z_2)\cdot\Im_2(z_1)+\Im_1(z_2)\cdot\Im_3(z_1))-\Im_2(z_2)\cdot\Re(z_1)-\Im_3(z_2)\cdot\Im_1(z_1)}{|z_2|^{\mathbf{2}}}, \frac{((\Re(z_2)\cdot\Im_3(z_1)-\Im_1(z_2)\cdot\Im_2(z_1))+\Im_2(z_2)\cdot\Im_1(z_1))-\Im_3(z_2)\cdot\Re(z_1)}{|z_2|^{\mathbf{2}}} \rangle_{\mathbb{H}}$.

(79) $(i)^{-1} = -i$.

(80) $(j)^{-1} = -j$.

(81) $(k)^{-1} = -k$.

Let z be a quaternion number. The functor $z^{\mathbf{2}}$ is defined by:

(Def. 1) $z^{\mathbf{2}} = z \cdot z$.

Let z be a quaternion number. One can verify that $z^{\mathbf{2}}$ is quaternion.

Let z be an element of $\mathbb{H}$. Then $z^{\mathbf{2}}$ is an element of $\mathbb{H}$.

One can prove the following four propositions:

(82) $z^{\mathbf{2}} = \langle (\Re(z))^{\mathbf{2}} - (\Im_1(z))^{\mathbf{2}} - (\Im_2(z))^{\mathbf{2}} - (\Im_3(z))^{\mathbf{2}}, 2\cdot(\Re(z)\cdot\Im_1(z)), 2\cdot(\Re(z)\cdot \Im_2(z)), 2\cdot(\Re(z)\cdot\Im_3(z))\rangle_{\mathbb{H}}$.

(83) $(0_{\mathbb{H}})^{\mathbf{2}} = 0$.

(84) $(1_{\mathbb{H}})^{\mathbf{2}} = 1$.

(85) $z^{\mathbf{2}} = (-z)^{\mathbf{2}}$.

Let z be a quaternion number. The functor $z^{\mathbf{3}}$ is defined as follows:

(Def. 2) $z^{\mathbf{3}} = z \cdot z \cdot z$.

Let z be a quaternion number. Observe that $z^{\mathbf{3}}$ is quaternion.

Let z be an element of $\mathbb{H}$. Then $z^{\mathbf{3}}$ is an element of $\mathbb{H}$.

Next we state several propositions:

(86) $(0_{\mathbb{H}})^{\mathbf{3}} = 0$.

(87) $(1_{\mathbb{H}})^{\mathbf{3}} = 1$.

(88) $(i)^{\mathbf{3}} = -i$.

(89) $(j)^{\mathbf{3}} = -j$.

(90) $(k)^{\mathbf{3}} = -k$.

(91) $(-1_{\mathbb{H}})^{\mathbf{2}} = 1$.

(92) $(-1_{\mathbb{H}})^{\mathbf{3}} = -1$.

(93) $z^{\mathbf{3}} = -(-z)^{\mathbf{3}}$.

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