

Some Operations on Quaternion Numbers

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Summary. In this article, we give some equality and basic theorems about quaternion numbers, and some special operations.

MML identifier: QUATERN3, version: 7.11.02 4.120.1050

The articles [11], [1], [12], [3], [4], [9], [2], [5], [8], [7], [10], [13], and [6] provide the notation and terminology for this paper.

In this paper z_1, z_2, z_3, z_4, z are quaternion numbers.

The following propositions are true:

- (1) $\Re(z_1 \cdot z_2) = \Re(z_2 \cdot z_1)$.
- (2) If z is a real number, then $z + z_3 = \Re(z) + \Re(z_3) + \Im_1(z_3) \cdot i + \Im_2(z_3) \cdot j + \Im_3(z_3) \cdot k$.
- (3) If z is a real number, then $z - z_3 = \langle \Re(z) - \Re(z_3), -\Im_1(z_3), -\Im_2(z_3), -\Im_3(z_3) \rangle_{\mathbb{H}}$.
- (4) If z is a real number, then $z \cdot z_3 = \langle \Re(z) \cdot \Re(z_3), \Re(z) \cdot \Im_1(z_3), \Re(z) \cdot \Im_2(z_3), \Re(z) \cdot \Im_3(z_3) \rangle_{\mathbb{H}}$.
- (5) If z is a real number, then $z \cdot i = \langle 0, \Re(z), 0, 0 \rangle_{\mathbb{H}}$.
- (6) If z is a real number, then $z \cdot j = \langle 0, 0, \Re(z), 0 \rangle_{\mathbb{H}}$.
- (7) If z is a real number, then $z \cdot k = \langle 0, 0, 0, \Re(z) \rangle_{\mathbb{H}}$.
- (8) $z - 0_{\mathbb{H}} = z$.

- (9) If z is a real number, then $z \cdot z_1 = z_1 \cdot z$.
- (10) If $\mathfrak{S}_1(z) = 0$ and $\mathfrak{S}_2(z) = 0$ and $\mathfrak{S}_3(z) = 0$, then $z = \Re(z)$.
- (11) $|z|^2 = (\Re(z))^2 + (\mathfrak{S}_1(z))^2 + (\mathfrak{S}_2(z))^2 + (\mathfrak{S}_3(z))^2$.
- (12) $|z|^2 = |z \cdot \bar{z}|$.
- (13) $|z|^2 = \Re(z \cdot \bar{z})$.
- (14) $2 \cdot \Re(z) = \Re(z + \bar{z})$.
- (15) If z is a real number, then $\overline{z \cdot z_1} = \bar{z} \cdot \bar{z}_1$.
- (16) $\overline{z_1 \cdot z_2} = \bar{z}_2 \cdot \bar{z}_1$.
- (17) $|z_1 \cdot z_2|^2 = |z_1|^2 \cdot |z_2|^2$.
- (18) $i \cdot z_1 - z_1 \cdot i = \langle 0, 0, -2 \cdot \mathfrak{S}_3(z_1), 2 \cdot \mathfrak{S}_2(z_1) \rangle_{\mathbb{H}}$.
- (19) $i \cdot z_1 + z_1 \cdot i = \langle -2 \cdot \mathfrak{S}_1(z_1), 2 \cdot \Re(z_1), 0, 0 \rangle_{\mathbb{H}}$.
- (20) $j \cdot z_1 - z_1 \cdot j = \langle 0, 2 \cdot \mathfrak{S}_3(z_1), 0, -2 \cdot \mathfrak{S}_1(z_1) \rangle_{\mathbb{H}}$.
- (21) $j \cdot z_1 + z_1 \cdot j = \langle -2 \cdot \mathfrak{S}_2(z_1), 0, 2 \cdot \Re(z_1), 0 \rangle_{\mathbb{H}}$.
- (22) $k \cdot z_1 - z_1 \cdot k = \langle 0, -2 \cdot \mathfrak{S}_2(z_1), 2 \cdot \mathfrak{S}_1(z_1), 0 \rangle_{\mathbb{H}}$.
- (23) $k \cdot z_1 + z_1 \cdot k = \langle -2 \cdot \mathfrak{S}_3(z_1), 0, 0, 2 \cdot \Re(z_1) \rangle_{\mathbb{H}}$.
- (24) $\Re(\frac{1}{|z|^2} \cdot \bar{z}) = \frac{1}{|z|^2} \cdot \Re(z)$.
- (25) $\mathfrak{S}_1(\frac{1}{|z|^2} \cdot \bar{z}) = -\frac{1}{|z|^2} \cdot \mathfrak{S}_1(z)$.
- (26) $\mathfrak{S}_2(\frac{1}{|z|^2} \cdot \bar{z}) = -\frac{1}{|z|^2} \cdot \mathfrak{S}_2(z)$.
- (27) $\mathfrak{S}_3(\frac{1}{|z|^2} \cdot \bar{z}) = -\frac{1}{|z|^2} \cdot \mathfrak{S}_3(z)$.
- (28) $\frac{1}{|z|^2} \cdot \bar{z} = \langle \frac{1}{|z|^2} \cdot \Re(z), -\frac{1}{|z|^2} \cdot \mathfrak{S}_1(z), -\frac{1}{|z|^2} \cdot \mathfrak{S}_2(z), -\frac{1}{|z|^2} \cdot \mathfrak{S}_3(z) \rangle_{\mathbb{H}}$.
- (29) $z \cdot (\frac{1}{|z|^2} \cdot \bar{z}) = \langle \frac{|z|^2}{|z|^2}, 0, 0, 0 \rangle_{\mathbb{H}}$.
- (30) $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \mathfrak{S}_1(z_1) \cdot \mathfrak{S}_1(z_2) - \mathfrak{S}_2(z_1) \cdot \mathfrak{S}_2(z_2) - \mathfrak{S}_3(z_1) \cdot \mathfrak{S}_3(z_2)$.
- (31) $\mathfrak{S}_1(z_1 \cdot z_2) = (\Re(z_1) \cdot \mathfrak{S}_1(z_2) + \mathfrak{S}_1(z_1) \cdot \Re(z_2) + \mathfrak{S}_2(z_1) \cdot \mathfrak{S}_3(z_2)) - \mathfrak{S}_3(z_1) \cdot \mathfrak{S}_2(z_2)$.
- (32) $\mathfrak{S}_2(z_1 \cdot z_2) = (\Re(z_1) \cdot \mathfrak{S}_2(z_2) + \mathfrak{S}_2(z_1) \cdot \Re(z_2) + \mathfrak{S}_3(z_1) \cdot \mathfrak{S}_1(z_2)) - \mathfrak{S}_1(z_1) \cdot \mathfrak{S}_3(z_2)$.
- (33) $\mathfrak{S}_3(z_1 \cdot z_2) = (\Re(z_1) \cdot \mathfrak{S}_3(z_2) + \mathfrak{S}_3(z_1) \cdot \Re(z_2) + \mathfrak{S}_1(z_1) \cdot \mathfrak{S}_2(z_2)) - \mathfrak{S}_2(z_1) \cdot \mathfrak{S}_1(z_2)$.
- (34) $|z_1 \cdot z_2 \cdot z_3|^2 = |z_1|^2 \cdot |z_2|^2 \cdot |z_3|^2$.
- (35) $\Re(z_1 \cdot z_2 \cdot z_3) = \Re(z_3 \cdot z_1 \cdot z_2)$.
- (36) $|z \cdot z| = |\bar{z} \cdot \bar{z}|$.
- (37) $|\bar{z} \cdot \bar{z}| = |z|^2$.
- (38) $|z_1 \cdot z_2 \cdot z_3| = |z_1| \cdot |z_2| \cdot |z_3|$.
- (39) $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$.
- (40) $|(z_1 + z_2) - z_3| \leq |z_1| + |z_2| + |z_3|$.
- (41) $|z_1 - z_2 - z_3| \leq |z_1| + |z_2| + |z_3|$.

- (42) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.
- (43) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.
- (44) $||z_1| - |z_2|| \leq |z_2 - z_1|$.
- (45) $||z_1| - |z_2|| \leq |z_1| + |z_2|$.
- (46) $|z_1| - |z_2| \leq |z_1 - z| + |z - z_2|$.
- (47) If $|z_1| - |z_2| \neq 0$, then $(|z_1|^2 + |z_2|^2) - 2 \cdot |z_1| \cdot |z_2| > 0$.
- (48) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.
- (49) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.
- (50) $(z_1 \cdot z_2)^{-1} = z_2^{-1} \cdot z_1^{-1}$.
- (51) $\bar{z}^{-1} = \overline{z^{-1}}$.
- (52) $(1_{\mathbb{H}})^{-1} = 1_{\mathbb{H}}$.
- (53) If $|z_1| = |z_2|$ and $|z_1| \neq 0$ and $z_1^{-1} = z_2^{-1}$, then $z_1 = z_2$.
- (54) $(z_1 - z_2) \cdot (z_3 + z_4) = ((z_1 \cdot z_3 - z_2 \cdot z_3) + z_1 \cdot z_4) - z_2 \cdot z_4$.
- (55) $(z_1 + z_2) \cdot (z_3 + z_4) = z_1 \cdot z_3 + z_2 \cdot z_3 + z_1 \cdot z_4 + z_2 \cdot z_4$.
- (56) $-(z_1 + z_2) = -z_1 - z_2$.
- (57) $-(z_1 - z_2) = -z_1 + z_2$.
- (58) $z - (z_1 + z_2) = z - z_1 - z_2$.
- (59) $z - (z_1 - z_2) = (z - z_1) + z_2$.
- (60) $(z_1 + z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 + z_2 \cdot z_3) - z_1 \cdot z_4 - z_2 \cdot z_4$.
- (61) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 - z_2 \cdot z_3 - z_1 \cdot z_4) + z_2 \cdot z_4$.
- (62) $-(z_1 + z_2 + z_3) = -z_1 - z_2 - z_3$.
- (63) $-(z_1 - z_2 - z_3) = -z_1 + z_2 + z_3$.
- (64) $-((z_1 - z_2) + z_3) = (-z_1 + z_2) - z_3$.
- (65) $-((z_1 + z_2) - z_3) = (-z_1 - z_2) + z_3$.
- (66) If $z_1 + z = z_2 + z$, then $z_1 = z_2$.
- (67) If $z_1 - z = z_2 - z$, then $z_1 = z_2$.
- (68) $((z_1 + z_2) - z_3) \cdot z_4 = (z_1 \cdot z_4 + z_2 \cdot z_4) - z_3 \cdot z_4$.
- (69) $((z_1 - z_2) + z_3) \cdot z_4 = (z_1 \cdot z_4 - z_2 \cdot z_4) + z_3 \cdot z_4$.
- (70) $(z_1 - z_2 - z_3) \cdot z_4 = z_1 \cdot z_4 - z_2 \cdot z_4 - z_3 \cdot z_4$.
- (71) $(z_1 + z_2 + z_3) \cdot z_4 = z_1 \cdot z_4 + z_2 \cdot z_4 + z_3 \cdot z_4$.
- (72) $(z_1 - z_2) \cdot z_3 = (z_2 - z_1) \cdot -z_3$.
- (73) $z_3 \cdot (z_1 - z_2) = (-z_3) \cdot (z_2 - z_1)$.
- (74) $(z_1 - z_2 - z_3) + z_4 = (z_4 - z_3 - z_2) + z_1$.
- (75) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_2 - z_1) \cdot (z_4 - z_3)$.
- (76) $z - z_1 - z_2 = z - z_2 - z_1$.
- (77) $z^{-1} = \left\langle \frac{\Re(z)}{|z|^2}, -\frac{\Im_1(z)}{|z|^2}, -\frac{\Im_2(z)}{|z|^2}, -\frac{\Im_3(z)}{|z|^2} \right\rangle_{\mathbb{H}}$.

$$(78) \quad \frac{z_1}{z_2} = \left(\frac{\Re(z_2) \cdot \Re(z_1) + \Im_1(z_1) \cdot \Im_1(z_2) + \Im_2(z_2) \cdot \Im_2(z_1) + \Im_3(z_2) \cdot \Im_3(z_1)}{|z_2|^2}, \right. \\ \left. \frac{(\Re(z_2) \cdot \Im_1(z_1) - \Im_1(z_2) \cdot \Re(z_1) - \Im_2(z_2) \cdot \Im_3(z_1)) + \Im_3(z_2) \cdot \Im_2(z_1)}{|z_2|^2}, \right. \\ \left. \frac{(\Re(z_2) \cdot \Im_2(z_1) + \Im_1(z_2) \cdot \Im_3(z_1)) - \Im_2(z_2) \cdot \Re(z_1) - \Im_3(z_2) \cdot \Im_1(z_1)}{|z_2|^2}, \right. \\ \left. \frac{((\Re(z_2) \cdot \Im_3(z_1) - \Im_1(z_2) \cdot \Im_2(z_1)) + \Im_2(z_2) \cdot \Im_1(z_1)) - \Im_3(z_2) \cdot \Re(z_1))}{|z_2|^2} \right)_{\mathbb{H}}.$$

$$(79) \quad (i)^{-1} = -i.$$

$$(80) \quad (j)^{-1} = -j.$$

$$(81) \quad (k)^{-1} = -k.$$

Let z be a quaternion number. The functor z^2 is defined by:

$$(\text{Def. 1}) \quad z^2 = z \cdot z.$$

Let z be a quaternion number. One can verify that z^2 is quaternion.

Let z be an element of \mathbb{H} . Then z^2 is an element of \mathbb{H} .

One can prove the following four propositions:

$$(82) \quad z^2 = \langle (\Re(z))^2 - (\Im_1(z))^2 - (\Im_2(z))^2 - (\Im_3(z))^2, 2 \cdot (\Re(z) \cdot \Im_1(z)), 2 \cdot (\Re(z) \cdot \Im_2(z)), 2 \cdot (\Re(z) \cdot \Im_3(z)) \rangle_{\mathbb{H}}.$$

$$(83) \quad (0_{\mathbb{H}})^2 = 0.$$

$$(84) \quad (1_{\mathbb{H}})^2 = 1.$$

$$(85) \quad z^2 = (-z)^2.$$

Let z be a quaternion number. The functor z^3 is defined as follows:

$$(\text{Def. 2}) \quad z^3 = z \cdot z \cdot z.$$

Let z be a quaternion number. Observe that z^3 is quaternion.

Let z be an element of \mathbb{H} . Then z^3 is an element of \mathbb{H} .

Next we state several propositions:

$$(86) \quad (0_{\mathbb{H}})^3 = 0.$$

$$(87) \quad (1_{\mathbb{H}})^3 = 1.$$

$$(88) \quad (i)^3 = -i.$$

$$(89) \quad (j)^3 = -j.$$

$$(90) \quad (k)^3 = -k.$$

$$(91) \quad (-1_{\mathbb{H}})^2 = 1.$$

$$(92) \quad (-1_{\mathbb{H}})^3 = -1.$$

$$(93) \quad z^3 = -(-z)^3.$$

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Received October 14, 2008
