

Complex Function Differentiability

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Summary. For a complex valued function defined on its domain in complex numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential complex functions are shown.

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The articles [17], [18], [3], [5], [4], [8], [2], [7], [11], [6], [16], [12], [19], [9], [10], [1], [14], [15], and [13] provide the notation and terminology for this paper.

For simplicity, we use the following convention: k, n, m denote elements of \mathbb{N} , X denotes a set, s_1, s_2 denote complex sequences, Y denotes a subset of \mathbb{C} , f, f_1, f_2 denote partial functions from \mathbb{C} to \mathbb{C} , r denotes a real number, $a, a_1, b, x, x_0, z, z_0$ denote complex numbers, and N_1 denotes an increasing sequence of naturals.

Let I be a complex sequence. We say that I is convergent to 0 if and only if:

(Def. 1) I is non-zero and convergent and $\lim I = 0$.

We now state four propositions:

- (1) Let r_1 be a sequence of real numbers and c_1 be a complex sequence. If $r_1 = c_1$ and r_1 is convergent, then c_1 is convergent.
- (2) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then s_1 is convergent.
- (3) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then $\lim s_1 = 0$.
- (4) If for every n holds $s_1(n) = \frac{1}{n+1}$, then s_1 is convergent and $\lim s_1 = 0$.

Let us observe that there exists a complex sequence which is convergent to 0.

Let us note that there exists a complex sequence which is constant.

Next we state four propositions:

- (5) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.
- (6) For every n holds $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.
- (7) If s_1 is constant and s_2 is a subsequence of s_1 , then s_2 is constant.
- (8) If s_1 is constant and s_2 is a subsequence of s_1 , then $s_1 = s_2$.

Let s_3 be a constant complex sequence. Note that every subsequence of s_3 is constant.

In the sequel h is a convergent to 0 complex sequence and c is a constant complex sequence.

Let I be a partial function from \mathbb{C} to \mathbb{C} . We say that I is rest-like if and only if:

- (Def. 2) I is total and for every h holds $h^{-1}(I \cdot h)$ is convergent and $\lim(h^{-1}(I \cdot h)) = 0$.

Let us mention that there exists a partial function from \mathbb{C} to \mathbb{C} which is rest-like.

A \mathbb{C} -rest is a rest-like partial function from \mathbb{C} to \mathbb{C} .

Let I be a partial function from \mathbb{C} to \mathbb{C} . We say that I is linear if and only if:

- (Def. 3) I is total and there exists a such that for every z holds $I_z = a \cdot z$.

One can check that there exists a partial function from \mathbb{C} to \mathbb{C} which is linear.

A \mathbb{C} -linear function is a linear partial function from \mathbb{C} to \mathbb{C} .

We adopt the following convention: R, R_1, R_2 are \mathbb{C} -rests and L, L_1, L_2 are \mathbb{C} -linear functions.

Let us consider L_1, L_2 . Observe that $L_1 + L_2$ is linear and $L_1 - L_2$ is linear.

The following propositions are true:

- (9) For all L_1, L_2 holds $L_1 + L_2$ is a \mathbb{C} -linear function and $L_1 - L_2$ is a \mathbb{C} -linear function.
- (10) For all a, L holds aL is a \mathbb{C} -linear function.
- (11) For all R_1, R_2 holds $R_1 + R_2$ is a \mathbb{C} -rest and $R_1 - R_2$ is a \mathbb{C} -rest and $R_1 R_2$ is a \mathbb{C} -rest.
- (12) aR is a \mathbb{C} -rest.
- (13) $L_1 L_2$ is rest-like.
- (14) RL is a \mathbb{C} -rest and LR is a \mathbb{C} -rest.

Let z_0 be a complex number. A subset of \mathbb{C} is called a neighbourhood of z_0 if:

(Def. 4) There exists a real number g such that $0 < g$ and $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\} \subseteq \text{it}$.

Next we state three propositions:

- (15) For every real number g such that $0 < g$ holds $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\}$ is a neighbourhood of z_0 .
- (16) For every neighbourhood N of z_0 holds $z_0 \in N$.
- (17) Let z_0 be a complex number and N_2, N_3 be neighbourhoods of z_0 . Then there exists a neighbourhood N of z_0 such that $N \subseteq N_2$ and $N \subseteq N_3$.

Let us consider f and let x_0 be a complex number. We say that f is differentiable in x_0 if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a neighbourhood N of x_0 such that $N \subseteq \text{dom } f$ and there exist L, R such that for every complex number x such that $x \in N$ holds $f_x - f_{x_0} = L_{x-x_0} + R_{x-x_0}$.

Let us consider f and let z_0 be a complex number. Let us assume that f is differentiable in z_0 . The functor $f'(z_0)$ yielding a complex number is defined by the condition (Def. 6).

(Def. 6) There exists a neighbourhood N of z_0 such that $N \subseteq \text{dom } f$ and there exist L, R such that $f'(z_0) = L_{1\mathbb{C}}$ and for every complex number z such that $z \in N$ holds $f_z - f_{z_0} = L_{z-z_0} + R_{z-z_0}$.

Let us consider f, X . We say that f is differentiable on X if and only if:

(Def. 7) $X \subseteq \text{dom } f$ and for every x such that $x \in X$ holds $f|X$ is differentiable in x .

We now state the proposition

- (18) If f is differentiable on X , then X is a subset of \mathbb{C} .

Let X be a subset of \mathbb{C} . We say that X is closed if and only if:

(Def. 8) For every complex sequence s_3 such that $\text{rng } s_3 \subseteq X$ and s_3 is convergent holds $\lim s_3 \in X$.

Let X be a subset of \mathbb{C} . We say that X is open if and only if:

(Def. 9) X^c is closed.

Next we state several propositions:

- (19) Let X be a subset of \mathbb{C} . Suppose X is open. Let z_0 be a complex number. If $z_0 \in X$, then there exists a neighbourhood N of z_0 such that $N \subseteq X$.
- (20) Let X be a subset of \mathbb{C} . Suppose X is open. Let z_0 be a complex number. Suppose $z_0 \in X$. Then there exists a real number g such that $\{y; y \text{ ranges over complex numbers: } |y - z_0| < g\} \subseteq X$.
- (21) Let X be a subset of \mathbb{C} . Suppose that for every complex number z_0 such that $z_0 \in X$ there exists a neighbourhood N of z_0 such that $N \subseteq X$. Then X is open.

- (22) Let X be a subset of \mathbb{C} . Then X is open if and only if for every complex number x such that $x \in X$ there exists a neighbourhood N of x such that $N \subseteq X$.
- (23) Let X be a subset of \mathbb{C} , z_0 be an element of \mathbb{C} , and r be an element of \mathbb{R} . If $X = \{y; y \text{ ranges over complex numbers: } |y - z_0| < r\}$, then X is open.
- (24) Let X be a subset of \mathbb{C} , z_0 be an element of \mathbb{C} , and r be an element of \mathbb{R} . If $X = \{y; y \text{ ranges over complex numbers: } |y - z_0| \leq r\}$, then X is closed.

Let us note that there exists a subset of \mathbb{C} which is open.

In the sequel Z denotes an open subset of \mathbb{C} .

Next we state two propositions:

- (25) f is differentiable on Z iff $Z \subseteq \text{dom } f$ and for every x such that $x \in Z$ holds f is differentiable in x .
- (26) If f is differentiable on Y , then Y is open.

Let us consider f, X . Let us assume that f is differentiable on X . The functor $f'_{\uparrow X}$ yielding a partial function from \mathbb{C} to \mathbb{C} is defined by:

(Def. 10) $\text{dom}(f'_{\uparrow X}) = X$ and for every x such that $x \in X$ holds $(f'_{\uparrow X})_x = f'(x)$.

The following propositions are true:

- (27) Let given f, Z . Suppose $Z \subseteq \text{dom } f$ and there exists a_1 such that $\text{rng } f = \{a_1\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\uparrow Z})_x = 0_{\mathbb{C}}$.
- (28) If s_1 is non-zero, then $s_1 \uparrow k$ is non-zero.

Let us consider h, n . Note that $h \uparrow n$ is convergent to 0.

Let us consider c, n . Note that $c \uparrow n$ is constant.

Next we state a number of propositions:

- (29) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.
- (30) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.
- (31) $s_1^{-1} \uparrow k = (s_1 \uparrow k)^{-1}$.
- (32) $(s_1 s_2) \uparrow k = (s_1 \uparrow k) (s_2 \uparrow k)$.
- (33) Let x_0 be a complex number and N be a neighbourhood of x_0 . Suppose f is differentiable in x_0 and $N \subseteq \text{dom } f$. Let given h, c . Suppose $\text{rng } c = \{x_0\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}(f \cdot (h + c) - f \cdot c)$ is convergent and $f'(x_0) = \lim(h^{-1}(f \cdot (h + c) - f \cdot c))$.
- (34) Let given f_1, f_2, x_0 . Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0 . Then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$.
- (35) Let given f_1, f_2, x_0 . Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0 . Then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$.

- (36) For all a, f, x_0 such that f is differentiable in x_0 holds $a f$ is differentiable in x_0 and $(a f)'(x_0) = a \cdot f'(x_0)$.
- (37) Let given f_1, f_2, x_0 . Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0 . Then $f_1 f_2$ is differentiable in x_0 and $(f_1 f_2)'(x_0) = (f_2)_{x_0} \cdot f_1'(x_0) + (f_1)_{x_0} \cdot f_2'(x_0)$.
- (38) For all f, Z such that $Z \subseteq \text{dom } f$ and $f \upharpoonright Z = \text{id}_Z$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f' \upharpoonright Z)_x = 1_{\mathbb{C}}$.
- (39) Let given f_1, f_2, Z . Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z . Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 + f_2)' \upharpoonright Z)_x = f_1'(x) + f_2'(x)$.
- (40) Let given f_1, f_2, Z . Suppose $Z \subseteq \text{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z . Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 - f_2)' \upharpoonright Z)_x = f_1'(x) - f_2'(x)$.
- (41) Let given a, f, Z . Suppose $Z \subseteq \text{dom}(a f)$ and f is differentiable on Z . Then $a f$ is differentiable on Z and for every x such that $x \in Z$ holds $((a f)' \upharpoonright Z)_x = a \cdot f'(x)$.
- (42) Let given f_1, f_2, Z . Suppose $Z \subseteq \text{dom}(f_1 f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z . Then $f_1 f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 f_2)' \upharpoonright Z)_x = (f_2)_x \cdot f_1'(x) + (f_1)_x \cdot f_2'(x)$.
- (43) If $Z \subseteq \text{dom } f$ and f is a constant on Z , then f is differentiable on Z and for every x such that $x \in Z$ holds $(f' \upharpoonright Z)_x = 0_{\mathbb{C}}$.
- (44) Suppose $Z \subseteq \text{dom } f$ and for every x such that $x \in Z$ holds $f_x = a \cdot x + b$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f' \upharpoonright Z)_x = a$.
- (45) For every complex number x_0 such that f is differentiable in x_0 holds f is continuous in x_0 .
- (46) If f is differentiable on X , then f is continuous on X .
- (47) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z .
- (48) If s_1 is convergent, then $|s_1|$ is convergent.
- (49) If f is differentiable in x_0 , then there exists R such that $R_{0_{\mathbb{C}}} = 0_{\mathbb{C}}$ and R is continuous in $0_{\mathbb{C}}$.

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