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Kolmogorov's Zero-One Law

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Summary. This article presents the proof of Kolmogorov's zero-one law in probability theory. The independence of a family of σ-fields is defined and basic theorems on it are given.

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The articles [8], [19], [2], [10], [12], [18], [20], [1], [15], [5], [21], [11], [3], [9], [7], [6], [17], [4], [16], [14], and [13] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: Ω, I are non empty sets, $\mathcal{F}$ is a σ-field of subsets of Ω, P is a probability on $\mathcal{F}$, D, E, F are families of subsets of Ω, A, B, s are non empty subsets of $\mathcal{F}$, b is an element of B, a is an element of $\mathcal{F}$, p, q, u, v are events of $\mathcal{F}$, n is an element of $\mathbb{N}$, and i is a set.

Next we state three propositions:

(1) For every function f and for every set X such that $X \subseteq \operatorname{dom} f$ holds if $X \neq \emptyset$, then $\operatorname{rng}(f{\restriction}X) \neq \emptyset$.

(2) For every real number r such that $r \cdot r = r$ holds $r = 0$ or $r = 1$.

(3) For every family X of subsets of Ω such that $X = \emptyset$ holds $\sigma(X) = \{\emptyset, \Omega\}$.

Let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let B be a subset of $\mathcal{F}$, and let P be a probability on $\mathcal{F}$. The functor $\operatorname{Indep}(B, P)$ yielding a subset of $\mathcal{F}$ is defined as follows:

(Def. 1) For every element a of $\mathcal{F}$ holds $a \in \operatorname{Indep}(B, P)$ iff for every element b of B holds $P(a \cap b) = P(a) \cdot P(b)$.

Next we state several propositions:

(4) Let f be a sequence of subsets of $\mathcal{F}$. Suppose for all n, b holds $P(f(n) \cap b) = P(f(n)) \cdot P(b)$ and f is disjoint valued. Then $P(b \cap \bigcup f) = P(b) \cdot P(\bigcup f)$.

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(5) $\mathrm{Indep}(B, P)$ is a Dynkin system of Ω.

(6) For every family A of subsets of Ω such that A is intersection stable and $A \subseteq \mathrm{Indep}(B, P)$ holds $\sigma(A) \subseteq \mathrm{Indep}(B, P)$.

(7) Let A, B be non empty subsets of $\mathcal{F}$. Then $A \subseteq \mathrm{Indep}(B, P)$ if and only if for all p, q such that $p \in A$ and $q \in B$ holds p and q are independent w.r.t. P.

(8) For all non empty subsets A, B of $\mathcal{F}$ such that $A \subseteq \mathrm{Indep}(B, P)$ holds $B \subseteq \mathrm{Indep}(A, P)$.

(9) Let A be a family of subsets of Ω. Suppose A is a non empty subset of $\mathcal{F}$ and intersection stable. Let B be a non empty subset of $\mathcal{F}$. Suppose B is intersection stable. If $A \subseteq \mathrm{Indep}(B, P)$, then for all D, s such that $D = B$ and $\sigma(D) = s$ holds $\sigma(A) \subseteq \mathrm{Indep}(s, P)$.

(10) Let given E, F. Suppose that

(i) E is a non empty subset of $\mathcal{F}$ and intersection stable, and

(ii) F is a non empty subset of $\mathcal{F}$ and intersection stable.

Suppose that for all p, q such that $p \in E$ and $q \in F$ holds p and q are independent w.r.t. P. Let given u, v. If $u \in \sigma(E)$ and $v \in \sigma(F)$, then u and v are independent w.r.t. P.

Let I be a set, let Ω be a non empty set, and let $\mathcal{F}$ be a σ-field of subsets of Ω. A function from I into $2^{\mathcal{F}}$ is said to be a many sorted σ-field over I and $\mathcal{F}$ if:

(Def. 2) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a σ-field of subsets of Ω.

Let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let P be a probability on $\mathcal{F}$, let I be a set, and let A be a function from I into $\mathcal{F}$. We say that A is independent w.r.t. P if and only if:

(Def. 3) For every one-to-one finite sequence e of elements of I such that $e \neq \emptyset$ holds $\prod(P \cdot A \cdot e) = P(\bigcap \mathrm{rng}(A \cdot e))$.

Let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let I be a set, let J be a subset of I, and let F be a many sorted σ-field over I and $\mathcal{F}$. A function from J into $\mathcal{F}$ is said to be a σ-section over J and F if:

(Def. 4) For every i such that $i \in J$ holds $\mathrm{it}(i) \in F(i)$.

Let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let P be a probability on $\mathcal{F}$, let I be a set, and let F be a many sorted σ-field over I and $\mathcal{F}$. We say that F is independent w.r.t. P if and only if:

(Def. 5) For every finite subset E of I holds every σ-section over E and F is independent w.r.t. P.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let F be a many sorted σ-field over I and $\mathcal{F}$, and let J be a subset of I. Then $F \restriction J$ is a function from J into $2^{\mathcal{F}}$.

Let I be a set, let J be a subset of I, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a function from J into $2^{\mathcal{F}}$. Then $\bigcup F$ is a family of subsets of Ω.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let F be a many sorted σ-field over I and $\mathcal{F}$, and let J be a subset of I. The functor $\mathrm{sigUn}(F, J)$ yields a σ-field of subsets of Ω and is defined as follows:

(Def. 6) $\mathrm{sigUn}(F, J) = \sigma(\bigcup(F \restriction J))$.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a many sorted σ-field over I and $\mathcal{F}$. The functor $\mathrm{futSigmaFields}(F, I)$ yielding a family of subsets of 2^{Ω} is defined as follows:

(Def. 7) For every family S of subsets of Ω holds $S \in \mathrm{futSigmaFields}(F, I)$ iff there exists a finite subset E of I such that $S = \mathrm{sigUn}(F, I \setminus E)$.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a many sorted σ-field over I and $\mathcal{F}$. Note that $\mathrm{futSigmaFields}(F, I)$ is non empty.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a many sorted σ-field over I and $\mathcal{F}$. The functor $\mathrm{tailSigmaField}(F, I)$ yielding a family of subsets of Ω is defined as follows:

(Def. 8) $\mathrm{tailSigmaField}(F, I) = \bigcap \mathrm{futSigmaFields}(F, I)$.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a many sorted σ-field over I and $\mathcal{F}$. Note that $\mathrm{tailSigmaField}(F, I)$ is non empty.

Let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, let I be a non empty set, let J be a non empty subset of I, and let F be a many sorted σ-field over I and $\mathcal{F}$. The functor $\mathrm{MeetSections}(J, F)$ yields a family of subsets of Ω and is defined by the condition (Def. 9).

(Def. 9) Let x be a subset of Ω. Then $x \in \mathrm{MeetSections}(J, F)$ if and only if there exists a non empty finite subset E of I and there exists a σ-section f over E and F such that $E \subseteq J$ and $x = \bigcap \mathrm{rng}\, f$.

One can prove the following propositions:

(11) For every many sorted σ-field F over I and $\mathcal{F}$ and for every non empty subset J of I holds $\sigma(\mathrm{MeetSections}(J, F)) = \mathrm{sigUn}(F, J)$.

(12) Let F be a many sorted σ-field over I and $\mathcal{F}$ and J, K be non empty subsets of I. Suppose F is independent w.r.t. P and J misses K. Let a, c be subsets of Ω. If $a \in \mathrm{MeetSections}(J, F)$ and $c \in \mathrm{MeetSections}(K, F)$, then $P(a \cap c) = P(a) \cdot P(c)$.

(13) Let F be a many sorted σ-field over I and $\mathcal{F}$ and J be a non empty subset of I. Then $\mathrm{MeetSections}(J, F)$ is a non empty subset of $\mathcal{F}$.

Let us consider I, Ω, $\mathcal{F}$, let F be a many sorted σ-field over I and $\mathcal{F}$, and let J be a non empty subset of I. Observe that $\mathrm{MeetSections}(J, F)$ is intersection stable.

The following proposition is true

(14) Let F be a many sorted σ-field over I and $\mathcal{F}$ and J, K be non empty subsets of I. Suppose F is independent w.r.t. P and J misses K. Let given u, v. If $u \in \mathrm{sigUn}(F, J)$ and $v \in \mathrm{sigUn}(F, K)$, then $P(u \cap v) = P(u) \cdot P(v)$.

Let I be a set, let Ω be a non empty set, let $\mathcal{F}$ be a σ-field of subsets of Ω, and let F be a many sorted σ-field over I and $\mathcal{F}$. The functor $\mathrm{finSigmaFields}(F, I)$ yielding a family of subsets of Ω is defined as follows:

(Def. 10) For every subset S of Ω holds $S \in \mathrm{finSigmaFields}(F, I)$ iff there exists a finite subset E of I such that $S \in \mathrm{sigUn}(F, E)$.

One can prove the following propositions:

(15) For every many sorted σ-field F over I and $\mathcal{F}$ holds $\mathrm{tailSigmaField}(F, I)$ is a σ-field of subsets of Ω.

(16) Let F be a many sorted σ-field over I and $\mathcal{F}$. If F is independent w.r.t. P and $a \in \mathrm{tailSigmaField}(F, I)$, then $P(a) = 0$ or $P(a) = 1$.

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