

Second-Order Partial Differentiation of Real Binary Functions

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Summary. In this article we define second-order partial differentiation of real binary functions and discuss the relation of second-order partial derivatives and partial derivatives defined in [17].

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The articles [15], [3], [4], [16], [5], [10], [1], [8], [11], [9], [2], [14], [6], [13], [12], [7], and [17] provide the notation and terminology for this paper.

1. SECOND-ORDER PARTIAL DERIVATIVES

For simplicity, we adopt the following convention: x, x_0, y, y_0, r are real numbers, z, z_0 are elements of \mathcal{R}^2 , f, f_1, f_2 are partial functions from \mathcal{R}^2 to \mathbb{R} , R is a rest, and L is a linear function.

Let us note that every rest is total.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . The functor $\text{pdiff1}(f, z)$ yielding a function from \mathcal{R}^2 into \mathbb{R} is defined as follows:

(Def. 1) For every z such that $z \in \mathcal{R}^2$ holds $(\text{pdiff1}(f, z))(z) = \text{partdiff1}(f, z)$.

The functor $\text{pdiff2}(f, z)$ yields a function from \mathcal{R}^2 into \mathbb{R} and is defined as follows:

(Def. 2) For every z such that $z \in \mathcal{R}^2$ holds $(\text{pdiff2}(f, z))(z) = \text{partdiff2}(f, z)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . We say that f is partial differentiable on 1st-1st coordinate in z if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff1}(f, z), z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff1}(f, z), z))(x) - (\text{SVF1}(\text{pdiff1}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 1st-2nd coordinate in z if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff1}(f, z), z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff1}(f, z), z))(y) - (\text{SVF2}(\text{pdiff1}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in z if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff2}(f, z), z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff2}(f, z), z))(x) - (\text{SVF1}(\text{pdiff2}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in z if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff2}(f, z), z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff2}(f, z), z))(y) - (\text{SVF2}(\text{pdiff2}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . Let us assume that f is partial differentiable on 1st-1st coordinate in z . The functor $\text{hpartdiff11}(f, z)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and

- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff1}(f, z), z)$ and there exist L, R such that $\text{hpartdiff11}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff1}(f, z), z))(x) - (\text{SVF1}(\text{pdiff1}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . Let us assume that f is partial differentiable on 1st-2nd coordinate in z . The functor $\text{hpartdiff12}(f, z)$ yielding a real number is defined by the condition (Def. 8).

(Def. 8) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff1}(f, z), z)$ and there exist L, R such that $\text{hpartdiff12}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff1}(f, z), z))(y) - (\text{SVF2}(\text{pdiff1}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . Let us assume that f is partial differentiable on 2nd-1st coordinate in z . The functor $\text{hpartdiff21}(f, z)$ yielding a real number is defined by the condition (Def. 9).

(Def. 9) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff2}(f, z), z)$ and there exist L, R such that $\text{hpartdiff21}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff2}(f, z), z))(x) - (\text{SVF1}(\text{pdiff2}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let z be an element of \mathcal{R}^2 . Let us assume that f is partial differentiable on 2nd-2nd coordinate in z . The functor $\text{hpartdiff22}(f, z)$ yields a real number and is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0 such that

- (i) $z = \langle x_0, y_0 \rangle$, and
(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff2}(f, z), z)$ and there exist L, R such that $\text{hpartdiff22}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff2}(f, z), z))(y) - (\text{SVF2}(\text{pdiff2}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Next we state several propositions:

- (1) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z , then $\text{SVF1}(\text{pdiff1}(f, z), z)$ is differentiable in x_0 .
- (2) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z , then $\text{SVF2}(\text{pdiff1}(f, z), z)$ is differentiable in y_0 .
- (3) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z , then $\text{SVF1}(\text{pdiff2}(f, z), z)$ is differentiable in x_0 .

- (4) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z , then $\text{SVF2}(\text{pdiff2}(f, z), z)$ is differentiable in y_0 .
- (5) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z , then $\text{hpartdiff11}(f, z) = (\text{SVF1}(\text{pdiff1}(f, z), z))'(x_0)$.
- (6) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z , then $\text{hpartdiff12}(f, z) = (\text{SVF2}(\text{pdiff1}(f, z), z))'(y_0)$.
- (7) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z , then $\text{hpartdiff21}(f, z) = (\text{SVF1}(\text{pdiff2}(f, z), z))'(x_0)$.
- (8) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z , then $\text{hpartdiff22}(f, z) = (\text{SVF2}(\text{pdiff2}(f, z), z))'(y_0)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let Z be a set. We say that f is partial differentiable on 1st-1st coordinate on Z if and only if:

- (Def. 11) $Z \subseteq \text{dom } f$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z$ is partial differentiable on 1st-1st coordinate in z .

We say that f is partial differentiable on 1st-2nd coordinate on Z if and only if:

- (Def. 12) $Z \subseteq \text{dom } f$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z$ is partial differentiable on 1st-2nd coordinate in z .

We say that f is partial differentiable on 2nd-1st coordinate on Z if and only if:

- (Def. 13) $Z \subseteq \text{dom } f$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z$ is partial differentiable on 2nd-1st coordinate in z .

We say that f is partial differentiable on 2nd-2nd coordinate on Z if and only if:

- (Def. 14) $Z \subseteq \text{dom } f$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z$ is partial differentiable on 2nd-2nd coordinate in z .

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let Z be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on Z . The functor $f|_Z^{\text{1st-1st}}$ yields a partial function from \mathcal{R}^2 to \mathbb{R} and is defined by:

- (Def. 15) $\text{dom}(f|_Z^{\text{1st-1st}}) = Z$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z^{\text{1st-1st}}(z) = \text{hpartdiff11}(f, z)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let Z be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on Z . The functor $f|_Z^{\text{1st-2nd}}$ yielding a partial function from \mathcal{R}^2 to \mathbb{R} is defined by:

- (Def. 16) $\text{dom}(f|_Z^{\text{1st-2nd}}) = Z$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f|_Z^{\text{1st-2nd}}(z) = \text{hpartdiff12}(f, z)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let Z be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on Z . The functor $f|_Z^{\text{2nd-1st}}$ yields a partial function from \mathcal{R}^2 to \mathbb{R} and is defined by:

(Def. 17) $\text{dom}(f_{|Z}^{2\text{nd}-1\text{st}}) = Z$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f_{|Z}^{2\text{nd}-1\text{st}}(z) = \text{hpartdiff21}(f, z)$.

Let f be a partial function from \mathcal{R}^2 to \mathbb{R} and let Z be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on Z . The functor $f_{|Z}^{2\text{nd}-2\text{nd}}$ yields a partial function from \mathcal{R}^2 to \mathbb{R} and is defined by:

(Def. 18) $\text{dom}(f_{|Z}^{2\text{nd}-2\text{nd}}) = Z$ and for every element z of \mathcal{R}^2 such that $z \in Z$ holds $f_{|Z}^{2\text{nd}-2\text{nd}}(z) = \text{hpartdiff22}(f, z)$.

2. MAIN PROPERTIES OF SECOND-ORDER PARTIAL DERIVATIVES

One can prove the following propositions:

- (9) f is partial differentiable on 1st-1st coordinate in z if and only if $\text{pdiff1}(f, z)$ is partial differentiable on 1st coordinate in z .
- (10) f is partial differentiable on 1st-2nd coordinate in z if and only if $\text{pdiff1}(f, z)$ is partial differentiable on 2nd coordinate in z .
- (11) f is partial differentiable on 2nd-1st coordinate in z if and only if $\text{pdiff2}(f, z)$ is partial differentiable on 1st coordinate in z .
- (12) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\text{pdiff2}(f, z)$ is partial differentiable on 2nd coordinate in z .
- (13) f is partial differentiable on 1st-1st coordinate in z if and only if $\text{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.
- (14) f is partial differentiable on 1st-2nd coordinate in z if and only if $\text{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.
- (15) f is partial differentiable on 2nd-1st coordinate in z if and only if $\text{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.
- (16) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\text{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.
- (17) If f is partial differentiable on 1st-1st coordinate in z , then $\text{hpartdiff11}(f, z) = \text{partdiff1}(\text{pdiff1}(f, z), z)$.
- (18) If f is partial differentiable on 1st-2nd coordinate in z , then $\text{hpartdiff12}(f, z) = \text{partdiff2}(\text{pdiff1}(f, z), z)$.
- (19) If f is partial differentiable on 2nd-1st coordinate in z , then $\text{hpartdiff21}(f, z) = \text{partdiff1}(\text{pdiff2}(f, z), z)$.
- (20) If f is partial differentiable on 2nd-2nd coordinate in z , then $\text{hpartdiff22}(f, z) = \text{partdiff2}(\text{pdiff2}(f, z), z)$.
- (21) Let z_0 be an element of \mathcal{R}^2 and N be a neighbourhood of $(\text{proj}(1, 2))(z_0)$. Suppose f is partial differentiable on 1st-1st coordinate in z_0 and $N \subseteq \text{dom SVF1}(\text{pdiff1}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c =$

$\{(\text{proj}(1, 2))(z_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}(\text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot (h + c) - \text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff11}(f, z_0) = \lim(h^{-1}(\text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot (h + c) - \text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot c))$.

(22) Let z_0 be an element of \mathcal{R}^2 and N be a neighbourhood of $(\text{proj}(2, 2))(z_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in z_0 and $N \subseteq \text{dom SVF2}(\text{pdiff1}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2, 2))(z_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}(\text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot (h + c) - \text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff12}(f, z_0) = \lim(h^{-1}(\text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot (h + c) - \text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot c))$.

(23) Let z_0 be an element of \mathcal{R}^2 and N be a neighbourhood of $(\text{proj}(1, 2))(z_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in z_0 and $N \subseteq \text{dom SVF1}(\text{pdiff2}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(1, 2))(z_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}(\text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot (h + c) - \text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff21}(f, z_0) = \lim(h^{-1}(\text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot (h + c) - \text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot c))$.

(24) Let z_0 be an element of \mathcal{R}^2 and N be a neighbourhood of $(\text{proj}(2, 2))(z_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in z_0 and $N \subseteq \text{dom SVF2}(\text{pdiff2}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2, 2))(z_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}(\text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot (h + c) - \text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff22}(f, z_0) = \lim(h^{-1}(\text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot (h + c) - \text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot c))$.

(25) Suppose that

- (i) f_1 is partial differentiable on 1st-1st coordinate in z_0 , and
- (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0 .

Then $\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff11}(f_1, z_0) + \text{hpartdiff11}(f_2, z_0)$.

(26) Suppose that

- (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0 , and
- (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0 .

Then $\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff12}(f_1, z_0) + \text{hpartdiff12}(f_2, z_0)$.

(27) Suppose that

- (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0 , and
- (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0 .

Then $\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0), z_0) = \text{hpartdiff21}(f_1, z_0) + \text{hpartdiff21}(f_2, z_0)$.

- hpartdiff21(f_2, z_0).
- (28) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0 .
- Then $\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0), z_0) = \text{hpartdiff22}(f_1, z_0) + \text{hpartdiff22}(f_2, z_0)$.
- (29) Suppose that
- (i) f_1 is partial differentiable on 1st-1st coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0 .
- Then $\text{pdiff1}(f_1, z_0) - \text{pdiff1}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff1}(f_1, z_0) - \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff11}(f_1, z_0) - \text{hpartdiff11}(f_2, z_0)$.
- (30) Suppose that
- (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0 .
- Then $\text{pdiff1}(f_1, z_0) - \text{pdiff1}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(\text{pdiff1}(f_1, z_0) - \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff12}(f_1, z_0) - \text{hpartdiff12}(f_2, z_0)$.
- (31) Suppose that
- (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0 .
- Then $\text{pdiff2}(f_1, z_0) - \text{pdiff2}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff2}(f_1, z_0) - \text{pdiff2}(f_2, z_0), z_0) = \text{hpartdiff21}(f_1, z_0) - \text{hpartdiff21}(f_2, z_0)$.
- (32) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0 .
- Then $\text{pdiff2}(f_1, z_0) - \text{pdiff2}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(\text{pdiff2}(f_1, z_0) - \text{pdiff2}(f_2, z_0), z_0) = \text{hpartdiff22}(f_1, z_0) - \text{hpartdiff22}(f_2, z_0)$.
- (33) Suppose f is partial differentiable on 1st-1st coordinate in z_0 . Then $r \text{pdiff1}(f, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(r \text{pdiff1}(f, z_0), z_0) = r \cdot \text{hpartdiff11}(f, z_0)$.
- (34) Suppose f is partial differentiable on 1st-2nd coordinate in z_0 . Then $r \text{pdiff1}(f, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(r \text{pdiff1}(f, z_0), z_0) = r \cdot \text{hpartdiff12}(f, z_0)$.
- (35) Suppose f is partial differentiable on 2nd-1st coordinate in z_0 . Then $r \text{pdiff2}(f, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(r \text{pdiff2}(f, z_0), z_0) = r \cdot \text{hpartdiff21}(f, z_0)$.

- (36) Suppose f is partial differentiable on 2nd-2nd coordinate in z_0 . Then $r \text{ pdiff2}(f, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(r \text{ pdiff2}(f, z_0), z_0) = r \cdot \text{hpartdiff22}(f, z_0)$.
- (37) Suppose that
- (i) f_1 is partial differentiable on 1st-1st coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0 .
- Then $\text{pdiff1}(f_1, z_0) \text{ pdiff1}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 .
- (38) Suppose that
- (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0 .
- Then $\text{pdiff1}(f_1, z_0) \text{ pdiff1}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 .
- (39) Suppose that
- (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0 .
- Then $\text{pdiff2}(f_1, z_0) \text{ pdiff2}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 .
- (40) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0 , and
 - (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0 .
- Then $\text{pdiff2}(f_1, z_0) \text{ pdiff2}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 .
- (41) Let z_0 be an element of \mathcal{R}^2 . Suppose f is partial differentiable on 1st-1st coordinate in z_0 . Then $\text{SVF1}(\text{pdiff1}(f, z_0), z_0)$ is continuous in $(\text{proj}(1, 2))(z_0)$.
- (42) Let z_0 be an element of \mathcal{R}^2 . Suppose f is partial differentiable on 1st-2nd coordinate in z_0 . Then $\text{SVF2}(\text{pdiff1}(f, z_0), z_0)$ is continuous in $(\text{proj}(2, 2))(z_0)$.
- (43) Let z_0 be an element of \mathcal{R}^2 . Suppose f is partial differentiable on 2nd-1st coordinate in z_0 . Then $\text{SVF1}(\text{pdiff2}(f, z_0), z_0)$ is continuous in $(\text{proj}(1, 2))(z_0)$.
- (44) Let z_0 be an element of \mathcal{R}^2 . Suppose f is partial differentiable on 2nd-2nd coordinate in z_0 . Then $\text{SVF2}(\text{pdiff2}(f, z_0), z_0)$ is continuous in $(\text{proj}(2, 2))(z_0)$.
- (45) If f is partial differentiable on 1st-1st coordinate in z_0 , then there exists R such that $R(0) = 0$ and R is continuous in 0.
- (46) If f is partial differentiable on 1st-2nd coordinate in z_0 , then there exists R such that $R(0) = 0$ and R is continuous in 0.

- (47) If f is partial differentiable on 2nd-1st coordinate in z_0 , then there exists R such that $R(0) = 0$ and R is continuous in 0.
- (48) If f is partial differentiable on 2nd-2nd coordinate in z_0 , then there exists R such that $R(0) = 0$ and R is continuous in 0.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [7] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces \mathcal{R}^n . *Formalized Mathematics*, 15(2):65–72, 2007, doi:10.2478/v10037-007-0008-5.
- [8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [10] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(4):781–786, 1990.
- [11] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [12] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(4):787–791, 1990.
- [13] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.
- [14] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [17] Bing Xie, Xiquan Liang, and Hongwei Li. Partial differentiation of real binary functions. *Formalized Mathematics*, 16(4):333–338, 2008, doi:10.2478/v10037-008-0041-z.

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