

# Solution of Cubic and Quartic Equations

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**Summary.** In this article, the principal  $n$ -th root of a complex number is defined, the Vieta's formulas for polynomial equations of degree 2, 3 and 4 are formalized. The solution of quadratic equations, the Cardan's solution of cubic equations and the Descartes-Euler solution of quartic equations in terms of their complex coefficients are also presented [5].

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The articles [11], [1], [4], [2], [10], [6], [8], [9], [12], [7], and [3] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

In this paper  $a, b$  are complex numbers.

The following propositions are true:

- (1)  $a \cdot a = a^2$ .
- (2)  $a \cdot a \cdot a = a^3$ .
- (3)  $a \cdot a \cdot a \cdot a = a^4$ .
- (4)  $(a - b)^2 = (a^2 - 2 \cdot a \cdot b) + b^2$ .
- (5)  $(a - b)^3 = ((a^3 - 3 \cdot a^2 \cdot b) + 3 \cdot b^2 \cdot a) - b^3$ .
- (6)  $(a - b)^4 = (((a^4 - 4 \cdot a^3 \cdot b) + 6 \cdot a^2 \cdot b^2) - 4 \cdot b^3 \cdot a) + b^4$ .

Let  $n$  be a natural number and let  $r$  be a real number. We introduce  $r^{1/n}$  as a synonym of  $\sqrt[n]{r}$ .

Let  $n$  be a non zero natural number and let  $z$  be a complex number. The functor  $\sqrt[n]{z}$  yields a complex number and is defined by:

(Def. 1)  $\sqrt[n]{z} = |z|^{1/n} \cdot (\cos(\frac{\text{Arg } z}{n}) + \sin(\frac{\text{Arg } z}{n}) \cdot i)$ .

In the sequel  $z$  denotes a complex number and  $n_0$  denotes a non zero natural number.

The following propositions are true:

- (7)  $\sqrt[n_0]{z^{n_0}} = z$ .
- (8) For every real number  $r$  such that  $r \geq 0$  holds  $\sqrt[n_0]{r} = r^{1/n_0}$ .
- (9) For every real number  $r$  such that  $r > 0$  holds  $\sqrt[n_0]{\frac{z}{r}} = \frac{\sqrt[n_0]{z}}{\sqrt[n_0]{r}}$ .
- (10)  $z^2 = a$  iff  $z = \sqrt[2]{a}$  or  $z = -\sqrt[2]{a}$ .

## 2. SOLUTION OF QUADRATIC EQUATIONS

In the sequel  $a_0, a_1, a_2, s_1, s_2$  are complex numbers.

Next we state two propositions:

- (11) If  $a_1 = -(s_1 + s_2)$  and  $a_0 = s_1 \cdot s_2$ , then  $z^2 + a_1 \cdot z + a_0 = 0$  iff  $z = s_1$  or  $z = s_2$ .
- (12) If  $a_2 \neq 0$ , then  $a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  iff  $z = -\frac{a_1}{2 \cdot a_2} + \frac{\sqrt[2]{\Delta(a_0, a_1, a_2)}}{2 \cdot a_2}$  or  $z = -\frac{a_1}{2 \cdot a_2} - \frac{\sqrt[2]{\Delta(a_0, a_1, a_2)}}{2 \cdot a_2}$ .

## 3. SOLUTION OF CUBIC EQUATIONS

In the sequel  $a_3, x, q, r, s, s_3$  are complex numbers.

The following four propositions are true:

- (13) Suppose  $z = x - \frac{a_2}{3}$  and  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$ . Then  $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if  $(x^3 + 3 \cdot q \cdot x) - 2 \cdot r = 0$ .
- (14) If  $a_2 = -(s_1 + s_2 + s_3)$  and  $a_1 = s_1 \cdot s_2 + s_1 \cdot s_3 + s_2 \cdot s_3$  and  $a_0 = -s_1 \cdot s_2 \cdot s_3$ , then  $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  iff  $z = s_1$  or  $z = s_2$  or  $z = s_3$ .
- (15) Suppose  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $q \neq 0$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s = \sqrt[2]{q^3 + r^2}$  and  $s_1 = \sqrt[3]{r + s}$  and  $s_2 = -\frac{q}{s_1}$ . Then  $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if one of the following conditions is satisfied:
- (i)  $z = (s_1 + s_2) - \frac{a_2}{3}$ , or
- (ii)  $z = \left(-\frac{s_1 + s_2}{2} - \frac{a_2}{3}\right) + \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$ , or
- (iii)  $z = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$ .
- (16) Suppose  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $q = 0$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s_1 = \sqrt[3]{2 \cdot r}$ . Then  $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if one of the following conditions is satisfied:
- (i)  $z = s_1 - \frac{a_2}{3}$ , or
- (ii)  $z = \left(-\frac{s_1}{2} - \frac{a_2}{3}\right) + \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$ , or
- (iii)  $z = -\frac{s_1}{2} - \frac{a_2}{3} - \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$ .

Let  $a_0, a_1, a_2$  be complex numbers. The functor  $\rho_1(a_0, a_1, a_2)$  yielding a complex number is defined by:

- (Def. 2)(i) There exist  $r, s_1$  such that  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s_1 = \sqrt[3]{2 \cdot r}$  and  $\rho_1(a_0, a_1, a_2) = s_1 - \frac{a_2}{3}$  if  $3 \cdot a_1 - a_2^2 = 0$ ,
- (ii) there exist  $q, r, s, s_1, s_2$  such that  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s = \sqrt[2]{q^3 + r^2}$  and  $s_1 = \sqrt[3]{r + s}$  and  $s_2 = -\frac{q}{s_1}$  and  $\rho_1(a_0, a_1, a_2) = (s_1 + s_2) - \frac{a_2}{3}$ , otherwise.

The functor  $\rho_2(a_0, a_1, a_2)$  yields a complex number and is defined by:

- (Def. 3)(i) There exist  $r, s_1$  such that  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s_1 = \sqrt[3]{2 \cdot r}$  and  $\rho_2(a_0, a_1, a_2) = (-\frac{s_1}{2} - \frac{a_2}{3}) + \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$  if  $3 \cdot a_1 - a_2^2 = 0$ ,
- (ii) there exist  $q, r, s, s_1, s_2$  such that  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s = \sqrt[2]{q^3 + r^2}$  and  $s_1 = \sqrt[3]{r + s}$  and  $s_2 = -\frac{q}{s_1}$  and  $\rho_2(a_0, a_1, a_2) = (-\frac{s_1 + s_2}{2} - \frac{a_2}{3}) + \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$ , otherwise.

The functor  $\rho_3(a_0, a_1, a_2)$  yields a complex number and is defined as follows:

- (Def. 4)(i) There exist  $r, s_1$  such that  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s_1 = \sqrt[3]{2 \cdot r}$  and  $\rho_3(a_0, a_1, a_2) = -\frac{s_1}{2} - \frac{a_2}{3} - \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$  if  $3 \cdot a_1 - a_2^2 = 0$ ,
- (ii) there exist  $q, r, s, s_1, s_2$  such that  $q = \frac{3 \cdot a_1 - a_2^2}{9}$  and  $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2^3 - 27 \cdot a_0}{54}$  and  $s = \sqrt[2]{q^3 + r^2}$  and  $s_1 = \sqrt[3]{r + s}$  and  $s_2 = -\frac{q}{s_1}$  and  $\rho_3(a_0, a_1, a_2) = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$ , otherwise.

We now state four propositions:

- (17)  $\rho_1(a_0, a_1, a_2) + \rho_2(a_0, a_1, a_2) + \rho_3(a_0, a_1, a_2) = -a_2$ .
- (18)  $\rho_1(a_0, a_1, a_2) \cdot \rho_2(a_0, a_1, a_2) + \rho_1(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) + \rho_2(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) = a_1$ .
- (19)  $\rho_1(a_0, a_1, a_2) \cdot \rho_2(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) = -a_0$ .
- (20) If  $a_3 \neq 0$ , then  $a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  iff  $z = \rho_1(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$  or  $z = \rho_2(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$  or  $z = \rho_3(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$ .

#### 4. SOLUTION OF QUARTIC EQUATIONS

In the sequel  $a_4, p, s_4$  are complex numbers.

One can prove the following propositions:

- (21) Suppose  $z = x - \frac{a_3}{4}$  and  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$ . Then  $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if  $x^4 + 4 \cdot p \cdot x^2 + 8 \cdot q \cdot x + 4 \cdot r = 0$ .
- (22) Suppose  $a_3 = -(s_1 + s_2 + s_3 + s_4)$  and  $a_2 = s_1 \cdot s_2 + s_1 \cdot s_3 + s_1 \cdot s_4 + s_2 \cdot s_3 + s_2 \cdot s_4 + s_3 \cdot s_4$  and  $a_1 = -(s_1 \cdot s_2 \cdot s_3 + s_1 \cdot s_2 \cdot s_4 + s_1 \cdot s_3 \cdot s_4 + s_2 \cdot s_3 \cdot s_4)$  and  $a_0 = s_1 \cdot s_2 \cdot s_3 \cdot s_4$ . Then  $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if  $z = s_1$  or  $z = s_2$  or  $z = s_3$  or  $z = s_4$ .

- (23) Suppose  $q \neq 0$  and  $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_3 = -\frac{q}{s_1 \cdot s_2}$ . Then  $z^4 + 4 \cdot p \cdot z^2 + 8 \cdot q \cdot z + 4 \cdot r = 0$  if and only if  $z = s_1 + s_2 + s_3$  or  $z = s_1 - s_2 - s_3$  or  $z = (-s_1 + s_2) - s_3$  or  $z = (-s_1 - s_2) + s_3$ .
- (24) Suppose that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $q \neq 0$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_3 = -\frac{q}{s_1 \cdot s_2}$ . Then  $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if  $z = (s_1 + s_2 + s_3) - \frac{a_3}{4}$  or  $z = s_1 - s_2 - s_3 - \frac{a_3}{4}$  or  $z = (-s_1 + s_2) - s_3 - \frac{a_3}{4}$  or  $z = ((-s_1 - s_2) + s_3) - \frac{a_3}{4}$ .
- (25) Suppose  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $q = 0$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{p^2 - r}$ . Then  $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if one of the following conditions is satisfied:
- (i)  $z = \sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$ , or
  - (ii)  $z = -\sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$ , or
  - (iii)  $z = \sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$ , or
  - (iv)  $z = -\sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$ .

Let  $a_0, a_1, a_2, a_3$  be complex numbers. The functor  $\rho_1(a_0, a_1, a_2, a_3)$  yielding a complex number is defined by:

- (Def. 5)(i) There exist  $p, r, s_1$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{p^2 - r}$  and  $\rho_1(a_0, a_1, a_2, a_3) = \sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$  if  $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3 = 0$ ,
- (ii) there exist  $p, q, r, s_1, s_2, s_3$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_3 = -\frac{q}{s_1 \cdot s_2}$  and  $\rho_1(a_0, a_1, a_2, a_3) = (s_1 + s_2 + s_3) - \frac{a_3}{4}$ , otherwise.

The functor  $\rho_2(a_0, a_1, a_2, a_3)$  yields a complex number and is defined as follows:

- (Def. 6)(i) There exist  $p, r, s_1$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{p^2 - r}$  and  $\rho_2(a_0, a_1, a_2, a_3) = -\sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$  if  $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3 = 0$ ,
- (ii) there exist  $p, q, r, s_1, s_2, s_3$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_3 = -\frac{q}{s_1 \cdot s_2}$  and  $\rho_2(a_0, a_1, a_2, a_3) = ((-s_1 - s_2) + s_3) - \frac{a_3}{4}$ , otherwise.

The functor  $\rho_3(a_0, a_1, a_2, a_3)$  yielding a complex number is defined as follows:

- (Def. 7)(i) There exist  $p, r, s_1$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{p^2 - r}$  and  $\rho_3(a_0, a_1, a_2, a_3) = \sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$  if  $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3 = 0$ ,
- (ii) there exist  $p, q, r, s_1, s_2, s_3$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and

$$q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64} \text{ and } r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024} \text{ and } s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)} \text{ and } s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)} \text{ and } s_3 = -\frac{q}{s_1 \cdot s_2} \text{ and } \rho_3(a_0, a_1, a_2, a_3) = (-s_1 + s_2) - s_3 - \frac{a_3}{4}, \text{ otherwise.}$$

The functor  $\rho_4(a_0, a_1, a_2, a_3)$  yields a complex number and is defined as follows:

- (Def. 8)(i) There exist  $p, r, s_1$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{p^2 - r}$  and  $\rho_4(a_0, a_1, a_2, a_3) = -\sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$  if  $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3 = 0$ ,
- (ii) there exist  $p, q, r, s_1, s_2, s_3$  such that  $p = \frac{8 \cdot a_2 - 3 \cdot a_3^2}{32}$  and  $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3^3}{64}$  and  $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3^2 \cdot a_2) - 3 \cdot a_3^4}{1024}$  and  $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$  and  $s_3 = -\frac{q}{s_1 \cdot s_2}$  and  $\rho_4(a_0, a_1, a_2, a_3) = s_1 - s_2 - s_3 - \frac{a_3}{4}$ , otherwise.

One can prove the following propositions:

- (26)  $\rho_1(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) + \rho_3(a_0, a_1, a_2, a_3) + \rho_4(a_0, a_1, a_2, a_3) = -a_3.$
- (27)  $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = a_2.$
- (28)  $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = -a_1.$
- (29)  $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = a_0.$
- (30) Suppose  $a_4 \neq 0$ . Then  $a_4 \cdot z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$  if and only if  $z = \rho_1\left(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}\right)$  or  $z = \rho_2\left(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}\right)$  or  $z = \rho_3\left(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}\right)$  or  $z = \rho_4\left(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}\right).$

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