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Probability on Finite Set and Real-Valued Random Variables

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In the various branches of science, probability and randomness provide us with useful theoretical frameworks. The *Formalized Mathematics* has already published some articles concerning the probability: [23], [24], [25], and [30]. In order to apply those articles, we shall give some theorems concerning the probability and the real-valued random variables to prepare for further studies.

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The articles [12], [28], [3], [14], [1], [18], [27], [9], [29], [11], [4], [21], [10], [2], [5], [6], [20], [25], [24], [30], [7], [16], [17], [19], [8], [15], [26], [13], and [22] provide the notation and terminology for this paper.

1. Probability on Finite Set

One can prove the following four propositions:

(1) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f{\restriction}E \,\mathrm{d}M$.

(2) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\mathbb{R}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f{\restriction}E \,\mathrm{d}M$.

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(3) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f{\restriction}E \, \mathrm{d}M \leq \overline{\mathbb{R}}(a) \cdot M(E)$.

(4) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\mathbb{R}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f{\restriction}E \, \mathrm{d}M \leq \overline{\mathbb{R}}(a) \cdot M(E)$.

2. Random Variables

For simplicity, we follow the rules: O is a non empty set, r is a real number, S is a σ-field of subsets of O, P is a probability on S, and E is a finite non empty set.

Let E be a non empty set. We introduce the trivial σ-field of E as a synonym of 2^E. Then the trivial σ-field of E is a σ-field of subsets of E.

Next we state a number of propositions:

(5) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then there exists a finite sequence F of separated subsets of the trivial σ-field of O and there exists a finite sequence s of elements of $\operatorname{dom} f$ such that
$\operatorname{dom} f = \bigcup \operatorname{rng} F$ and $\operatorname{dom} F = \operatorname{dom} s$ and s is one-to-one and $\operatorname{rng} s = \operatorname{dom} f$ and $\operatorname{len} s = \overline{\overline{\operatorname{dom} f}}$ and for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = \{s(k)\}$ and for every natural number n and for all elements x, y of O such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.

(6) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then

(i) f is simple function in the trivial σ-field of O, and

(ii) $\operatorname{dom} f$ is an element of the trivial σ-field of O.

(7) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, and f be a partial function from O to $\mathbb{R}$. If $\operatorname{dom} f \neq \emptyset$ and $M(\operatorname{dom} f) < +\infty$, then f is integrable on M.

(8) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then there exists an element X of the trivial σ-field of O such that $\operatorname{dom} f = X$ and f is measurable on X.

(9) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\overline{\mathbb{R}}$, and s be a finite sequence of elements of O. Suppose $M(O) < +\infty$ and

s is one-to-one and $\operatorname{rng} s = O$ and $\operatorname{len} s = \overline{\overline{O}}$. Then there exists a finite sequence F of separated subsets of the trivial σ-field of O and there exists a finite sequence a of elements of $\mathbb{R}$ such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} a = \operatorname{dom} s$,

(iii) $\operatorname{dom} F = \operatorname{dom} s$,

(iv) for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = \{s(k)\}$, and

(v) for every natural number n and for all elements x, y of O such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.

(10) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\overline{\mathbb{R}}$, and s be a finite sequence of elements of O. Suppose that

(i) $M(O) < +\infty$,

(ii) $\operatorname{len} x = \overline{\overline{O}}$,

(iii) s is one-to-one,

(iv) $\operatorname{rng} s = O$,

(v) $\operatorname{len} s = \overline{\overline{O}}$, and

(vi) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$.

Then $\int f \, \mathrm{d}M = \sum x$.

(11) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, and f be a function from O into $\mathbb{R}$. Suppose $M(O) < +\infty$. Then there exists a finite sequence x of elements of $\overline{\mathbb{R}}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} x = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$, and

(vi) $\int f \, \mathrm{d}M = \sum x$.

(12) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose that

(i) $\operatorname{len} x = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$, and

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = f(s(n)) \cdot P(\{s(n)\})$.

Then $\int f \,\mathrm{d}\,\mathrm{P2M}\,P = \sum x$.

(13) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and f be a function from O into $\mathbb{R}$. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\mathrm{len}\,F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\mathrm{rng}\,s = O$,

(iv) $\mathrm{len}\,s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \mathrm{dom}\,F$ holds $F(n) = f(s(n)) \cdot P(\{s(n)\})$, and

(vi) $\int f \,\mathrm{d}\,\mathrm{P2M}\,P = \sum F$.

(14) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-increasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $A(N) = A(m)$.

(15) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-increasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $\mathrm{Intersection}\,A = A(m)$.

(16) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-decreasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $A(N) = A(m)$.

(17) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-decreasing. Then there exists a natural number N such that for every natural number m such that $N \leq m$ holds $\bigcup A = A(m)$.

Let us consider E. The trivial probability of E yielding a probability on the trivial σ-field of E is defined as follows:

(Def. 1) For every event A_1 of E holds (the trivial probability of E)$(A_1) = \mathrm{P}(A_1)$.

Let us consider O, S. A function from O into $\mathbb{R}$ is said to be a real-valued random variable of S if:

(Def. 2) There exists an element X of S such that $X = O$ and it is measurable on X.

In the sequel f, g are real-valued random variables of S.

Next we state the proposition

(18) $f + g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f + g$ is a real-valued random variable of S.

We now state the proposition

(19) $f - g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f - g$ is a real-valued random variable of S.

Next we state the proposition

(20) For every real number r holds $r\,f$ is a real-valued random variable of S.

Let us consider O, S, f and let r be a real number. Then $r\,f$ is a real-valued random variable of S.

Next we state two propositions:

(21) For all partial functions f, g from O to $\mathbb{R}$ holds $\overline{\mathbb{R}}(f)\,\overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f\,g)$.

(22) $f\,g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f\,g$ is a real-valued random variable of S.

Next we state two propositions:

(23) For every real number r such that $0 \leq r$ and f is non-negative holds f^r is a real-valued random variable of S.

(24) $|f|$ is a real-valued random variable of S.

Let us consider O, S, f. Then $|f|$ is a real-valued random variable of S.

We now state the proposition

(25) For every real number r such that $0 \leq r$ holds $|f|^r$ is a real-valued random variable of S.

Let us consider O, S, f, P. We say that f is integrable on P if and only if:

(Def. 3) f is integrable on $\mathrm{P2M}\,P$.

Let us consider O, S, P and let f be a real-valued random variable of S. Let us assume that f is integrable on P. The functor $E_P\{f\}$ yielding an element of $\mathbb{R}$ is defined as follows:

(Def. 4) $E_P\{f\} = \int f \,\mathrm{d}\,\mathrm{P2M}\,P$.

One can prove the following propositions:

(26) If f is integrable on P and g is integrable on P, then $E_P\{f+g\} = E_P\{f\} + E_P\{g\}$.

(27) If f is integrable on P, then $E_P\{r\,f\} = r \cdot E_P\{f\}$.

(28) If f is integrable on P and g is integrable on P, then $E_P\{f-g\} = E_P\{f\} - E_P\{g\}$.

(29) For every non empty finite set O holds every function from O into $\mathbb{R}$ is a real-valued random variable of the trivial σ-field of O.

(30) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then X is integrable on P.

(31) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, X be a real-valued random variable of the trivial σ-field of O, F be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose that

(i) $\mathrm{len}\,F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\mathrm{rng}\,s = O$,

(iv) $\mathrm{len}\,s = \overline{\overline{O}}$, and

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$.

Then $E_P\{X\} = \sum F$.

(32) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and

(vi) $E_P\{X\} = \sum F$.

(33) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and

(vi) $E_P\{X\} = \sum F$.

(34) Let O be a non empty finite set, X be a real-valued random variable of the trivial σ-field of O, G be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose $\operatorname{len} G = \overline{\overline{O}}$ and s is one-to-one and $\operatorname{rng} s = O$ and $\operatorname{len} s = \overline{\overline{O}}$ and for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = X(s(n))$. Then $E_{\text{the trivial probability of } O}\{X\} = \frac{\sum G}{\overline{\overline{O}}}$.

(35) Let O be a non empty finite set and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence G of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} G = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = X(s(n))$, and

(vi) $E_{\text{the trivial probability of } O}\{X\} = \frac{\sum G}{\overline{\overline{O}}}$.

(36) Let X be a real-valued random variable of S. Suppose $0 < r$ and X is non-negative and X is integrable on P. Then $P(\{t \in O\colon r \leq X(t)\}) \leq \frac{E_P\{X\}}{r}$.

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