

# Probability on Finite Set and Real-Valued Random Variables

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**Summary.** In the various branches of science, probability and randomness provide us with useful theoretical frameworks. The *Formalized Mathematics* has already published some articles concerning the probability: [23], [24], [25], and [30]. In order to apply those articles, we shall give some theorems concerning the probability and the real-valued random variables to prepare for further studies.

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The articles [12], [28], [3], [14], [1], [18], [27], [9], [29], [11], [4], [21], [10], [2], [5], [6], [20], [25], [24], [30], [7], [16], [17], [19], [8], [15], [26], [13], and [22] provide the notation and terminology for this paper.

## 1. PROBABILITY ON FINITE SET

One can prove the following four propositions:

- (1) Let  $X$  be a non empty set,  $S_1$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S_1$ ,  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ ,  $E$  be an element of  $S_1$ , and  $a$  be a real number. Suppose  $f$  is integrable on  $M$  and  $E \subseteq \text{dom } f$  and  $M(E) < +\infty$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $a \leq f(x)$ . Then  $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f \upharpoonright E \, dM$ .
- (2) Let  $X$  be a non empty set,  $S_1$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S_1$ ,  $f$  be a partial function from  $X$  to  $\mathbb{R}$ ,  $E$  be an element of  $S_1$ , and  $a$  be a real number. Suppose  $f$  is integrable on  $M$  and  $E \subseteq \text{dom } f$  and  $M(E) < +\infty$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $a \leq f(x)$ . Then  $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f \upharpoonright E \, dM$ .

- (3) Let  $X$  be a non empty set,  $S_1$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S_1$ ,  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ ,  $E$  be an element of  $S_1$ , and  $a$  be a real number. Suppose  $f$  is integrable on  $M$  and  $E \subseteq \text{dom } f$  and  $M(E) < +\infty$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $f(x) \leq a$ . Then  $\int f \mathbb{1}_E dM \leq \overline{\mathbb{R}}(a) \cdot M(E)$ .
- (4) Let  $X$  be a non empty set,  $S_1$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S_1$ ,  $f$  be a partial function from  $X$  to  $\mathbb{R}$ ,  $E$  be an element of  $S_1$ , and  $a$  be a real number. Suppose  $f$  is integrable on  $M$  and  $E \subseteq \text{dom } f$  and  $M(E) < +\infty$  and for every element  $x$  of  $X$  such that  $x \in E$  holds  $f(x) \leq a$ . Then  $\int f \mathbb{1}_E dM \leq \overline{\mathbb{R}}(a) \cdot M(E)$ .

## 2. RANDOM VARIABLES

For simplicity, we follow the rules:  $O$  is a non empty set,  $r$  is a real number,  $S$  is a  $\sigma$ -field of subsets of  $O$ ,  $P$  is a probability on  $S$ , and  $E$  is a finite non empty set.

Let  $E$  be a non empty set. We introduce the trivial  $\sigma$ -field of  $E$  as a synonym of  $2^E$ . Then the trivial  $\sigma$ -field of  $E$  is a  $\sigma$ -field of subsets of  $E$ .

Next we state a number of propositions:

- (5) Let  $O$  be a non empty finite set and  $f$  be a partial function from  $O$  to  $\mathbb{R}$ . Then there exists a finite sequence  $F$  of separated subsets of the trivial  $\sigma$ -field of  $O$  and there exists a finite sequence  $s$  of elements of  $\text{dom } f$  such that
- $\text{dom } f = \bigcup \text{rng } F$  and  $\text{dom } F = \text{dom } s$  and  $s$  is one-to-one and  $\text{rng } s = \text{dom } f$  and  $\text{len } s = \overline{\text{dom } f}$  and for every natural number  $k$  such that  $k \in \text{dom } F$  holds  $F(k) = \{s(k)\}$  and for every natural number  $n$  and for all elements  $x, y$  of  $O$  such that  $n \in \text{dom } F$  and  $x, y \in F(n)$  holds  $f(x) = f(y)$ .
- (6) Let  $O$  be a non empty finite set and  $f$  be a partial function from  $O$  to  $\mathbb{R}$ . Then
- (i)  $f$  is simple function in the trivial  $\sigma$ -field of  $O$ , and
  - (ii)  $\text{dom } f$  is an element of the trivial  $\sigma$ -field of  $O$ .
- (7) Let  $O$  be a non empty finite set,  $M$  be a  $\sigma$ -measure on the trivial  $\sigma$ -field of  $O$ , and  $f$  be a partial function from  $O$  to  $\mathbb{R}$ . If  $\text{dom } f \neq \emptyset$  and  $M(\text{dom } f) < +\infty$ , then  $f$  is integrable on  $M$ .
- (8) Let  $O$  be a non empty finite set and  $f$  be a partial function from  $O$  to  $\mathbb{R}$ . Then there exists an element  $X$  of the trivial  $\sigma$ -field of  $O$  such that  $\text{dom } f = X$  and  $f$  is measurable on  $X$ .
- (9) Let  $O$  be a non empty finite set,  $M$  be a  $\sigma$ -measure on the trivial  $\sigma$ -field of  $O$ ,  $f$  be a function from  $O$  into  $\mathbb{R}$ ,  $x$  be a finite sequence of elements of  $\overline{\mathbb{R}}$ , and  $s$  be a finite sequence of elements of  $O$ . Suppose  $M(O) < +\infty$  and

$s$  is one-to-one and  $\text{rng } s = O$  and  $\text{len } s = \overline{O}$ . Then there exists a finite sequence  $F$  of separated subsets of the trivial  $\sigma$ -field of  $O$  and there exists a finite sequence  $a$  of elements of  $\mathbb{R}$  such that

- (i)  $\text{dom } f = \bigcup \text{rng } F$ ,
  - (ii)  $\text{dom } a = \text{dom } s$ ,
  - (iii)  $\text{dom } F = \text{dom } s$ ,
  - (iv) for every natural number  $k$  such that  $k \in \text{dom } F$  holds  $F(k) = \{s(k)\}$ ,  
and
  - (v) for every natural number  $n$  and for all elements  $x, y$  of  $O$  such that  $n \in \text{dom } F$  and  $x, y \in F(n)$  holds  $f(x) = f(y)$ .
- (10) Let  $O$  be a non empty finite set,  $M$  be a  $\sigma$ -measure on the trivial  $\sigma$ -field of  $O$ ,  $f$  be a function from  $O$  into  $\mathbb{R}$ ,  $x$  be a finite sequence of elements of  $\overline{\mathbb{R}}$ , and  $s$  be a finite sequence of elements of  $O$ . Suppose that
- (i)  $M(O) < +\infty$ ,
  - (ii)  $\text{len } x = \overline{O}$ ,
  - (iii)  $s$  is one-to-one,
  - (iv)  $\text{rng } s = O$ ,
  - (v)  $\text{len } s = \overline{O}$ , and
  - (vi) for every natural number  $n$  such that  $n \in \text{dom } x$  holds  $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$ .
- Then  $\int f \, dM = \sum x$ .

- (11) Let  $O$  be a non empty finite set,  $M$  be a  $\sigma$ -measure on the trivial  $\sigma$ -field of  $O$ , and  $f$  be a function from  $O$  into  $\mathbb{R}$ . Suppose  $M(O) < +\infty$ . Then there exists a finite sequence  $x$  of elements of  $\overline{\mathbb{R}}$  and there exists a finite sequence  $s$  of elements of  $O$  such that
- (i)  $\text{len } x = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ ,
  - (v) for every natural number  $n$  such that  $n \in \text{dom } x$  holds  $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$ , and
  - (vi)  $\int f \, dM = \sum x$ .

- (12) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ ,  $f$  be a function from  $O$  into  $\mathbb{R}$ ,  $x$  be a finite sequence of elements of  $\overline{\mathbb{R}}$ , and  $s$  be a finite sequence of elements of  $O$ . Suppose that
- (i)  $\text{len } x = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ , and
  - (v) for every natural number  $n$  such that  $n \in \text{dom } x$  holds  $x(n) = f(s(n)) \cdot P(\{s(n)\})$ .

Then  $\int f \, dP = \sum x$ .

- (13) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ , and  $f$  be a function from  $O$  into  $\mathbb{R}$ . Then there exists a finite sequence  $F$  of elements of  $\mathbb{R}$  and there exists a finite sequence  $s$  of elements of  $O$  such that
- (i)  $\text{len } F = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ ,
  - (v) for every natural number  $n$  such that  $n \in \text{dom } F$  holds  $F(n) = f(s(n)) \cdot P(\{s(n)\})$ , and
  - (vi)  $\int f \, dP = \sum F$ .
- (14) Let  $E$  be a finite non empty set and  $A$  be a sequence of subsets of  $E$ . Suppose  $A$  is non-increasing. Then there exists an element  $N$  of  $\mathbb{N}$  such that for every element  $m$  of  $\mathbb{N}$  such that  $N \leq m$  holds  $A(N) = A(m)$ .
- (15) Let  $E$  be a finite non empty set and  $A$  be a sequence of subsets of  $E$ . Suppose  $A$  is non-increasing. Then there exists an element  $N$  of  $\mathbb{N}$  such that for every element  $m$  of  $\mathbb{N}$  such that  $N \leq m$  holds  $\text{Intersection } A = A(m)$ .
- (16) Let  $E$  be a finite non empty set and  $A$  be a sequence of subsets of  $E$ . Suppose  $A$  is non-decreasing. Then there exists an element  $N$  of  $\mathbb{N}$  such that for every element  $m$  of  $\mathbb{N}$  such that  $N \leq m$  holds  $A(N) = A(m)$ .
- (17) Let  $E$  be a finite non empty set and  $A$  be a sequence of subsets of  $E$ . Suppose  $A$  is non-decreasing. Then there exists a natural number  $N$  such that for every natural number  $m$  such that  $N \leq m$  holds  $\bigcup A = A(m)$ .

Let us consider  $E$ . The trivial probability of  $E$  yielding a probability on the trivial  $\sigma$ -field of  $E$  is defined as follows:

(Def. 1) For every event  $A_1$  of  $E$  holds (the trivial probability of  $E$ )( $A_1$ ) =  $P(A_1)$ .

Let us consider  $O, S$ . A function from  $O$  into  $\mathbb{R}$  is said to be a real-valued random variable of  $S$  if:

(Def. 2) There exists an element  $X$  of  $S$  such that  $X = O$  and it is measurable on  $X$ .

In the sequel  $f, g$  are real-valued random variables of  $S$ .

Next we state the proposition

(18)  $f + g$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f, g$ . Then  $f + g$  is a real-valued random variable of  $S$ .

We now state the proposition

(19)  $f - g$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f, g$ . Then  $f - g$  is a real-valued random variable of  $S$ .

Next we state the proposition

(20) For every real number  $r$  holds  $r f$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f$  and let  $r$  be a real number. Then  $rf$  is a real-valued random variable of  $S$ .

Next we state two propositions:

- (21) For all partial functions  $f, g$  from  $O$  to  $\mathbb{R}$  holds  $\overline{\mathbb{R}}(f) \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(fg)$ .
- (22)  $fg$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f, g$ . Then  $fg$  is a real-valued random variable of  $S$ .

Next we state two propositions:

- (23) For every real number  $r$  such that  $0 \leq r$  and  $f$  is non-negative holds  $f^r$  is a real-valued random variable of  $S$ .
- (24)  $|f|$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f$ . Then  $|f|$  is a real-valued random variable of  $S$ .

We now state the proposition

- (25) For every real number  $r$  such that  $0 \leq r$  holds  $|f|^r$  is a real-valued random variable of  $S$ .

Let us consider  $O, S, f, P$ . We say that  $f$  is integrable on  $P$  if and only if:

(Def. 3)  $f$  is integrable on P2MP.

Let us consider  $O, S, P$  and let  $f$  be a real-valued random variable of  $S$ . Let us assume that  $f$  is integrable on  $P$ . The functor  $E_P\{f\}$  yielding an element of  $\mathbb{R}$  is defined as follows:

(Def. 4)  $E_P\{f\} = \int f \, dP2MP$ .

One can prove the following propositions:

- (26) If  $f$  is integrable on  $P$  and  $g$  is integrable on  $P$ , then  $E_P\{f + g\} = E_P\{f\} + E_P\{g\}$ .
- (27) If  $f$  is integrable on  $P$ , then  $E_P\{rf\} = r \cdot E_P\{f\}$ .
- (28) If  $f$  is integrable on  $P$  and  $g$  is integrable on  $P$ , then  $E_P\{f - g\} = E_P\{f\} - E_P\{g\}$ .
- (29) For every non empty finite set  $O$  holds every function from  $O$  into  $\mathbb{R}$  is a real-valued random variable of the trivial  $\sigma$ -field of  $O$ .
- (30) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ , and  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ . Then  $X$  is integrable on  $P$ .
- (31) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ ,  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ ,  $F$  be a finite sequence of elements of  $\mathbb{R}$ , and  $s$  be a finite sequence of elements of  $O$ . Suppose that
  - (i)  $\text{len } F = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ , and

- (v) for every natural number  $n$  such that  $n \in \text{dom } F$  holds  $F(n) = X(s(n)) \cdot P(\{s(n)\})$ .  
Then  $E_P\{X\} = \sum F$ .
- (32) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ , and  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ . Then there exists a finite sequence  $F$  of elements of  $\mathbb{R}$  and there exists a finite sequence  $s$  of elements of  $O$  such that
- (i)  $\text{len } F = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ ,
  - (v) for every natural number  $n$  such that  $n \in \text{dom } F$  holds  $F(n) = X(s(n)) \cdot P(\{s(n)\})$ , and
  - (vi)  $E_P\{X\} = \sum F$ .
- (33) Let  $O$  be a non empty finite set,  $P$  be a probability on the trivial  $\sigma$ -field of  $O$ , and  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ . Then there exists a finite sequence  $F$  of elements of  $\mathbb{R}$  and there exists a finite sequence  $s$  of elements of  $O$  such that
- (i)  $\text{len } F = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ ,
  - (v) for every natural number  $n$  such that  $n \in \text{dom } F$  holds  $F(n) = X(s(n)) \cdot P(\{s(n)\})$ , and
  - (vi)  $E_P\{X\} = \sum F$ .
- (34) Let  $O$  be a non empty finite set,  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ ,  $G$  be a finite sequence of elements of  $\mathbb{R}$ , and  $s$  be a finite sequence of elements of  $O$ . Suppose  $\text{len } G = \overline{O}$  and  $s$  is one-to-one and  $\text{rng } s = O$  and  $\text{len } s = \overline{O}$  and for every natural number  $n$  such that  $n \in \text{dom } G$  holds  $G(n) = X(s(n))$ . Then  $E_{\text{the trivial probability of } O}\{X\} = \frac{\sum G}{\overline{O}}$ .
- (35) Let  $O$  be a non empty finite set and  $X$  be a real-valued random variable of the trivial  $\sigma$ -field of  $O$ . Then there exists a finite sequence  $G$  of elements of  $\mathbb{R}$  and there exists a finite sequence  $s$  of elements of  $O$  such that
- (i)  $\text{len } G = \overline{O}$ ,
  - (ii)  $s$  is one-to-one,
  - (iii)  $\text{rng } s = O$ ,
  - (iv)  $\text{len } s = \overline{O}$ ,
  - (v) for every natural number  $n$  such that  $n \in \text{dom } G$  holds  $G(n) = X(s(n))$ ,  
and

- (vi)  $E_{\text{the trivial probability of } O\{X\}} = \frac{\sum O}{O}$ .
- (36) Let  $X$  be a real-valued random variable of  $S$ . Suppose  $0 < r$  and  $X$  is non-negative and  $X$  is integrable on  $P$ . Then  $P(\{t \in O: r \leq X(t)\}) \leq \frac{E_P\{X\}}{r}$ .

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