

Lebesgue's Convergence Theorem of Complex-Valued Function

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Summary. In this article, we formalized Lebesgue's Convergence theorem of complex-valued function. We proved Lebesgue's Convergence Theorem of real-valued function using the theorem of extensional real-valued function. Then applying the former theorem to real part and imaginary part of complex-valued functional sequences, we proved Lebesgue's Convergence Theorem of complex-valued function. We also defined partial sums of real-valued functional sequences and complex-valued functional sequences and showed their properties. In addition, we proved properties of complex-valued simple functions.

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The articles [24], [1], [4], [12], [25], [5], [26], [6], [7], [18], [19], [2], [8], [14], [13], [20], [21], [3], [11], [22], [15], [10], [16], [9], [17], and [23] provide the notation and terminology for this paper.

1. PARTIAL SUMS OF REAL-VALUED FUNCTIONAL SEQUENCES

For simplicity, we use the following convention: X is a non empty set, S is a σ -field of subsets of X , M is a σ -measure on S , E is an element of S , F is a sequence of partial functions from X into \mathbb{R} , f is a partial function from X to \mathbb{R} , s is a sequence of real numbers, n, m are natural numbers, x is an element of X , and z, D are sets.

Let X, Y be sets, let F be a sequence of partial functions from X into Y , and let D be a set. The functor $F \upharpoonright D$ yielding a sequence of partial functions from X into Y is defined by:

(Def. 1) For every natural number n holds $(F \upharpoonright D)(n) = F(n) \upharpoonright D$.

One can prove the following propositions:

- (1) If $x \in D$ and $F \# x$ is convergent, then $(F \upharpoonright D) \# x$ is convergent.
- (2) Let X, Y, D be sets and F be a sequence of partial functions from X into Y . If F has the same dom, then $F \upharpoonright D$ has the same dom.
- (3) If $D \subseteq \text{dom } F(0)$ and for every element x of X such that $x \in D$ holds $F \# x$ is convergent, then $\lim F \upharpoonright D = \lim(F \upharpoonright D)$.
- (4) Suppose F has the same dom and $E \subseteq \text{dom } F(0)$ and for every natural number m holds $F(m)$ is measurable on E . Then $(F \upharpoonright E)(n)$ is measurable on E .
- (5) $(\sum_{\alpha=0}^{\kappa} (\overline{\mathbb{R}}(s))(\alpha))_{\kappa \in \mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$.
- (6) Suppose that for every element x of X such that $x \in E$ holds $F \# x$ is summable. Let x be an element of X . If $x \in E$, then $(F \upharpoonright E) \# x$ is summable.

Let X be a non empty set and let F be a sequence of partial functions from X into \mathbb{R} . The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ yields a sequence of partial functions from X into \mathbb{R} and is defined by:

(Def. 2) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(0) = F(0)$ and for every element n of \mathbb{N} holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) + F(n+1)$.

One can prove the following propositions:

- (7) $(\sum_{\alpha=0}^{\kappa} (\overline{\mathbb{R}}(F))(\alpha))_{\kappa \in \mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$.
- (8) If $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $m \leq n$, then $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $z \in \text{dom } F(m)$.
- (9) $\overline{\mathbb{R}}(F)$ is additive.
- (10) $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \bigcap \{ \text{dom } F(k); k \text{ ranges over elements of } \mathbb{N}: k \leq n \}$.
- (11) If F has the same dom, then $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \text{dom } F(0)$.
- (12) If F has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)(n)$.
- (13) If F has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$ is convergent.
- (14) If F has the same dom and $\text{dom } f \subseteq \text{dom } F(0)$ and $x \in \text{dom } f$ and $f(x) = \sum(F \# x)$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)$.
- (15) If for every natural number m holds $F(m)$ is simple function in S , then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is simple function in S .

- (16) If for every natural number n holds $F(n)$ is measurable on E , then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is measurable on E .
- (17) Let X be a non empty set and F be a sequence of partial functions from X into \mathbb{R} . If F has the same dom, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ has the same dom.
- (18) Suppose that
- (i) $\text{dom } F(0) = E$,
 - (ii) F has the same dom,
 - (iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is measurable on E , and
 - (iv) for every element x of X such that $x \in E$ holds $F \# x$ is summable. Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$ is measurable on E .
- (19) Suppose that for every natural number n holds $F(n)$ is integrable on M . Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is integrable on M .

2. PARTIAL SUMS OF COMPLEX-VALUED FUNCTIONAL SEQUENCES

In the sequel F denotes a sequence of partial functions from X into \mathbb{C} , f denotes a partial function from X to \mathbb{C} , and A denotes a set.

We now state several propositions:

- (20) $\Re(f) \upharpoonright A = \Re(f \upharpoonright A)$ and $\Im(f) \upharpoonright A = \Im(f \upharpoonright A)$.
- (21) $\Re(F \upharpoonright D) = \Re(F) \upharpoonright D$.
- (22) $\Im(F \upharpoonright D) = \Im(F) \upharpoonright D$.
- (23) If F has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then if $F \# x$ is convergent, then $(F \upharpoonright D) \# x$ is convergent.
- (24) F has the same dom iff $\Re(F)$ has the same dom.
- (25) $\Re(F)$ has the same dom iff $\Im(F)$ has the same dom.
- (26) If F has the same dom and $D = \text{dom } F(0)$ and for every element x of X such that $x \in D$ holds $F \# x$ is convergent, then $\lim F \upharpoonright D = \lim(F \upharpoonright D)$.
- (27) Suppose F has the same dom and $E \subseteq \text{dom } F(0)$ and for every natural number m holds $F(m)$ is measurable on E . Then $(F \upharpoonright E)(n)$ is measurable on E .
- (28) Suppose $E \subseteq \text{dom } F(0)$ and F has the same dom and for every element x of X such that $x \in E$ holds $F \# x$ is summable. Let x be an element of X . If $x \in E$, then $(F \upharpoonright E) \# x$ is summable.

Let X be a non empty set and let F be a sequence of partial functions from X into \mathbb{C} . The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ yielding a sequence of partial functions from X into \mathbb{C} is defined by:

(Def. 3) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(0) = F(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) + F(n+1)$.

The following propositions are true:

- (29) $(\sum_{\alpha=0}^{\kappa} \Re(F)(\alpha))_{\kappa \in \mathbb{N}} = \Re((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa})$, $(\sum_{\alpha=0}^{\kappa} \Im(F)(\alpha))_{\kappa \in \mathbb{N}} = \Im((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$.
- (30) If $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $m \leq n$, then $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $z \in \text{dom } F(m)$.
- (31) $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \bigcap \{\text{dom } F(k); k \text{ ranges over elements of } \mathbb{N}; k \leq n\}$.
- (32) If F has the same dom, then $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \text{dom } F(0)$.
- (33) If F has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)(n)$.
- (34) If F has the same dom, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ has the same dom.
- (35) If F has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$ is convergent.
- (36) If F has the same dom and $\text{dom } f \subseteq \text{dom } F(0)$ and $x \in \text{dom } f$ and $F \# x$ is summable and $f(x) = \sum(F \# x)$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)$.
- (37) If for every natural number m holds $F(m)$ is simple function in S , then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is simple function in S .
- (38) If for every natural number n holds $F(n)$ is measurable on E , then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is measurable on E .
- (39) Suppose that
- (i) $\text{dom } F(0) = E$,
 - (ii) F has the same dom,
 - (iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is measurable on E , and
 - (iv) for every element x of X such that $x \in E$ holds $F \# x$ is summable. Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$ is measurable on E .
- (40) Suppose that for every natural number n holds $F(n)$ is integrable on M . Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is integrable on M .

3. SELECTED PROPERTIES OF COMPLEX-VALUED SIMPLE FUNCTIONS

In the sequel f, g denote partial functions from X to \mathbb{C} and A denotes an element of S .

Next we state several propositions:

- (41) If f is simple function in S , then f is measurable on A .
- (42) If f is simple function in S , then $f|A$ is simple function in S .

- (43) If f is simple function in S , then $\text{dom } f$ is an element of S .
- (44) If f is simple function in S and g is simple function in S , then $f + g$ is simple function in S .
- (45) For every complex number c such that f is simple function in S holds cf is simple function in S .

4. LEBESGUE'S CONVERGENCE THEOREM OF COMPLEX-VALUED FUNCTION

In the sequel F denotes a sequence of partial functions from X into $\overline{\mathbb{R}}$ with the same dom and P denotes a partial function from X to $\overline{\mathbb{R}}$.

The following proposition is true

- (46) Suppose that
- (i) $E = \text{dom } F(0)$,
 - (ii) $E = \text{dom } P$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) P is integrable on M ,
 - (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
 - (vi) for every element x of X such that $x \in E$ holds $F \# x$ is convergent.
- Then $\lim F$ is integrable on M .

In the sequel F denotes a sequence of partial functions from X into \mathbb{R} with the same dom and f, P denote partial functions from X to \mathbb{R} .

One can prove the following propositions:

- (47) Suppose that
- (i) $E = \text{dom } F(0)$,
 - (ii) $E = \text{dom } P$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) P is integrable on M ,
 - (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
 - (vi) for every element x of X such that $x \in E$ holds $F \# x$ is convergent.
- Then $\lim F$ is integrable on M .

- (48) Suppose that
- (i) $E = \text{dom } F(0)$,
 - (ii) $E = \text{dom } P$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) P is integrable on M , and
 - (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.
- Then there exists a sequence I of real numbers such that

- (vi) for every natural number n holds $I(n) = \int F(n) \, dM$, and
- (vii) if for every element x of X such that $x \in E$ holds $F \# x$ is convergent, then I is convergent and $\lim I = \int \lim F \, dM$.

Let X be a set and let F be a sequence of partial functions from X into \mathbb{R} . We say that F is uniformly bounded if and only if the condition (Def. 4) is satisfied.

- (Def. 4) There exists a real number K such that for every natural number n and for every element x of X if $x \in \text{dom } F(0)$, then $|F(n)(x)| \leq K$.

Next we state the proposition

- (49) Suppose that
- (i) $M(E) < +\infty$,
 - (ii) $E = \text{dom } F(0)$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) F is uniformly bounded, and
 - (v) for every element x of X such that $x \in E$ holds $F \# x$ is convergent.

Then

- (vi) for every natural number n holds $F(n)$ is integrable on M ,
- (vii) $\lim F$ is integrable on M , and
- (viii) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \, dM$ and I is convergent and $\lim I = \int \lim F \, dM$.

Let X be a set, let F be a sequence of partial functions from X into \mathbb{R} , and let f be a partial function from X to \mathbb{R} . We say that F is uniformly convergent to f if and only if the conditions (Def. 5) are satisfied.

- (Def. 5)(i) F has the same dom,
- (ii) $\text{dom } F(0) = \text{dom } f$, and
 - (iii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n and for every element x of X such that $n \geq N$ and $x \in \text{dom } F(0)$ holds $|F(n)(x) - f(x)| < e$.

We now state the proposition

- (50) Suppose that
- (i) $M(E) < +\infty$,
 - (ii) $E = \text{dom } F(0)$,
 - (iii) for every natural number n holds $F(n)$ is integrable on M , and
 - (iv) F is uniformly convergent to f .

Then

- (v) f is integrable on M , and
- (vi) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \, dM$ and I is convergent and $\lim I = \int f \, dM$.

In the sequel F denotes a sequence of partial functions from X into \mathbb{C} with the same dom and f denotes a partial function from X to \mathbb{C} .

The following two propositions are true:

- (51) Suppose that
- (i) $E = \text{dom } F(0)$,
 - (ii) $E = \text{dom } P$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) P is integrable on M ,
 - (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
 - (vi) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then $\lim F$ is integrable on M .

- (52) Suppose that
- (i) $E = \text{dom } F(0)$,
 - (ii) $E = \text{dom } P$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) P is integrable on M , and
 - (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then there exists a complex sequence I such that

- (vi) for every natural number n holds $I(n) = \int F(n) \, dM$, and
- (vii) if for every element x of X such that $x \in E$ holds $F\#x$ is convergent, then I is convergent and $\lim I = \int \lim F \, dM$.

Let X be a set and let F be a sequence of partial functions from X into \mathbb{C} . We say that F is uniformly bounded if and only if the condition (Def. 6) is satisfied.

- (Def. 6) There exists a real number K such that for every natural number n and for every element x of X if $x \in \text{dom } F(0)$, then $|F(n)(x)| \leq K$.

The following proposition is true

- (53) Suppose that
- (i) $M(E) < +\infty$,
 - (ii) $E = \text{dom } F(0)$,
 - (iii) for every natural number n holds $F(n)$ is measurable on E ,
 - (iv) F is uniformly bounded, and
 - (v) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then

- (vi) for every natural number n holds $F(n)$ is integrable on M ,
- (vii) $\lim F$ is integrable on M , and
- (viii) there exists a complex sequence I such that for every natural number n holds $I(n) = \int F(n) \, dM$ and I is convergent and $\lim I = \int \lim F \, dM$.

Let X be a set, let F be a sequence of partial functions from X into \mathbb{C} , and let f be a partial function from X to \mathbb{C} . We say that F is uniformly convergent to f if and only if the conditions (Def. 7) are satisfied.

- (Def. 7)(i) F has the same dom,
 (ii) $\text{dom } F(0) = \text{dom } f$, and
 (iii) for every real number ϵ such that $\epsilon > 0$ there exists a natural number N such that for every natural number n and for every element x of X such that $n \geq N$ and $x \in \text{dom } F(0)$ holds $|F(n)(x) - f(x)| < \epsilon$.

We now state the proposition

- (54) Suppose that
 (i) $M(E) < +\infty$,
 (ii) $E = \text{dom } F(0)$,
 (iii) for every natural number n holds $F(n)$ is integrable on M , and
 (iv) F is uniformly convergent to f .

Then

- (v) f is integrable on M , and
 (vi) there exists a complex sequence I such that for every natural number n holds $I(n) = \int F(n) \, dM$ and I is convergent and $\lim I = \int f \, dM$.

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