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Lebesgue's Convergence Theorem of Complex-Valued Function

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Summary. In this article, we formalized Lebesgue's Convergence theorem of complex-valued function. We proved Lebesgue's Convergence Theorem of real-valued function using the theorem of extensional real-valued function. Then applying the former theorem to real part and imaginary part of complex-valued functional sequences, we proved Lebesgue's Convergence Theorem of complex-valued function. We also defined partial sums of real-valued functional sequences and complex-valued functional sequences and showed their properties. In addition, we proved properties of complex-valued simple functions.

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The articles [24], [1], [4], [12], [25], [5], [26], [6], [7], [18], [19], [2], [8], [14], [13], [20], [21], [3], [11], [22], [15], [10], [16], [9], [17], and [23] provide the notation and terminology for this paper.

1. Partial Sums of Real-Valued Functional Sequences

For simplicity, we use the following convention: X is a non empty set, S is a σ-field of subsets of X, M is a σ-measure on S, E is an element of S, F is a sequence of partial functions from X into $\mathbb{R}$, f is a partial function from X to $\mathbb{R}$, s is a sequence of real numbers, n, m are natural numbers, x is an element of X, and z, D are sets.

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Let X, Y be sets, let F be a sequence of partial functions from X into Y, and let D be a set. The functor $F \restriction D$ yielding a sequence of partial functions from X into Y is defined by:

(Def. 1) For every natural number n holds $(F \restriction D)(n) = F(n){\restriction}D$.

One can prove the following propositions:

(1) If $x \in D$ and $F\#x$ is convergent, then $(F \restriction D)\#x$ is convergent.

(2) Let X, Y, D be sets and F be a sequence of partial functions from X into Y. If F has the same dom, then $F \restriction D$ has the same dom.

(3) If $D \subseteq \operatorname{dom} F(0)$ and for every element x of X such that $x \in D$ holds $F\#x$ is convergent, then $\lim F{\restriction}D = \lim(F \restriction D)$.

(4) Suppose F has the same dom and $E \subseteq \operatorname{dom} F(0)$ and for every natural number m holds $F(m)$ is measurable on E. Then $(F \restriction E)(n)$ is measurable on E.

(5) $(\sum_{\alpha=0}^{\kappa}(\overline{\mathbb{R}}(s))(\alpha))_{\kappa\in\mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})$.

(6) Suppose that for every element x of X such that $x \in E$ holds $F\#x$ is summable. Let x be an element of X. If $x \in E$, then $(F \restriction E)\#x$ is summable.

Let X be a non empty set and let F be a sequence of partial functions from X into $\mathbb{R}$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of partial functions from X into $\mathbb{R}$ and is defined by:

(Def. 2) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) + F(n+1)$.

One can prove the following propositions:

(7) $(\sum_{\alpha=0}^{\kappa}(\overline{\mathbb{R}}(F))(\alpha))_{\kappa\in\mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})$.

(8) If $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $m \leq n$, then $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ and $z \in \operatorname{dom} F(m)$.

(9) $\overline{\mathbb{R}}(F)$ is additive.

(10) $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \bigcap\{\operatorname{dom} F(k); k$ ranges over elements of $\mathbb{N}$: $k \leq n\}$.

(11) If F has the same dom, then $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \operatorname{dom} F(0)$.

(12) If F has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x)(n)$.

(13) If F has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent.

(14) If F has the same dom and $\operatorname{dom} f \subseteq \operatorname{dom} F(0)$ and $x \in \operatorname{dom} f$ and $f(x) = \sum(F\#x)$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x)$.

(15) If for every natural number m holds $F(m)$ is simple function in S, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is simple function in S.

(16) If for every natural number n holds $F(n)$ is measurable on E, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ is measurable on E.

(17) Let X be a non empty set and F be a sequence of partial functions from X into $\mathbb{R}$. If F has the same dom, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ has the same dom.

(18) Suppose that

(i) $\operatorname{dom} F(0) = E$,

(ii) F has the same dom,

(iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is measurable on E, and

(iv) for every element x of X such that $x \in E$ holds $F\#x$ is summable.

Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})$ is measurable on E.

(19) Suppose that for every natural number n holds $F(n)$ is integrable on M. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ is integrable on M.

2. Partial Sums of Complex-Valued Functional Sequences

In the sequel F denotes a sequence of partial functions from X into $\mathbb{C}$, f denotes a partial function from X to $\mathbb{C}$, and A denotes a set.

We now state several propositions:

(20) $\Re(f){\restriction}A = \Re(f{\restriction}A)$ and $\Im(f){\restriction}A = \Im(f{\restriction}A)$.

(21) $\Re(F \restriction D) = \Re(F) \restriction D$.

(22) $\Im(F \restriction D) = \Im(F) \restriction D$.

(23) If F has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then if $F\#x$ is convergent, then $(F \restriction D)\#x$ is convergent.

(24) F has the same dom iff $\Re(F)$ has the same dom.

(25) $\Re(F)$ has the same dom iff $\Im(F)$ has the same dom.

(26) If F has the same dom and $D = \operatorname{dom} F(0)$ and for every element x of X such that $x \in D$ holds $F\#x$ is convergent, then $\lim F{\restriction}D = \lim(F \restriction D)$.

(27) Suppose F has the same dom and $E \subseteq \operatorname{dom} F(0)$ and for every natural number m holds $F(m)$ is measurable on E. Then $(F \restriction E)(n)$ is measurable on E.

(28) Suppose $E \subseteq \operatorname{dom} F(0)$ and F has the same dom and for every element x of X such that $x \in E$ holds $F\#x$ is summable. Let x be an element of X. If $x \in E$, then $(F \restriction E)\#x$ is summable.

Let X be a non empty set and let F be a sequence of partial functions from X into $\mathbb{C}$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ yielding a sequence of partial functions from X into $\mathbb{C}$ is defined by:

(Def. 3) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(0) = F(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) + F(n+1)$.

The following propositions are true:

(29) $(\sum_{\alpha=0}^{\kappa} \Re(F)(\alpha))_{\kappa\in\mathbb{N}} = \Re((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa}$, $(\sum_{\alpha=0}^{\kappa} \Im(F)(\alpha))_{\kappa\in\mathbb{N}} = \Im((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})$.

(30) If $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $m \leq n$, then $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ and $z \in \operatorname{dom} F(m)$.

(31) $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \bigcap\{\operatorname{dom} F(k); k$ ranges over elements of $\mathbb{N}$: $k \leq n\}$.

(32) If F has the same dom, then $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \operatorname{dom} F(0)$.

(33) If F has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x)(n)$.

(34) If F has the same dom, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ has the same dom.

(35) If F has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent.

(36) If F has the same dom and $\operatorname{dom} f \subseteq \operatorname{dom} F(0)$ and $x \in \operatorname{dom} f$ and $F\#x$ is summable and $f(x) = \sum(F\#x)$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x)$.

(37) If for every natural number m holds $F(m)$ is simple function in S, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is simple function in S.

(38) If for every natural number n holds $F(n)$ is measurable on E, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ is measurable on E.

(39) Suppose that

(i) $\operatorname{dom} F(0) = E$,

(ii) F has the same dom,

(iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is measurable on E, and

(iv) for every element x of X such that $x \in E$ holds $F\#x$ is summable.

Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})$ is measurable on E.

(40) Suppose that for every natural number n holds $F(n)$ is integrable on M. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ is integrable on M.

3. Selected Properties of Complex-Valued Simple Functions

In the sequel f, g denote partial functions from X to $\mathbb{C}$ and A denotes an element of S.

Next we state several propositions:

(41) If f is simple function in S, then f is measurable on A.

(42) If f is simple function in S, then $f\restriction A$ is simple function in S.

(43) If f is simple function in S, then $\operatorname{dom} f$ is an element of S.

(44) If f is simple function in S and g is simple function in S, then $f+g$ is simple function in S.

(45) For every complex number c such that f is simple function in S holds $c\,f$ is simple function in S.

4. Lebesgue's Convergence Theorem of Complex-Valued Function

In the sequel F denotes a sequence of partial functions from X into $\overline{\mathbb{R}}$ with the same dom and P denotes a partial function from X to $\overline{\mathbb{R}}$.

The following proposition is true

(46) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) $E = \operatorname{dom} P$,
- (iii) for every natural number n holds $F(n)$ is measurable on E,
- (iv) P is integrable on M,
- (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
- (vi) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then $\lim F$ is integrable on M.

In the sequel F denotes a sequence of partial functions from X into $\mathbb{R}$ with the same dom and f, P denote partial functions from X to $\mathbb{R}$.

One can prove the following propositions:

(47) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) $E = \operatorname{dom} P$,
- (iii) for every natural number n holds $F(n)$ is measurable on E,
- (iv) P is integrable on M,
- (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
- (vi) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then $\lim F$ is integrable on M.

(48) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) $E = \operatorname{dom} P$,
- (iii) for every natural number n holds $F(n)$ is measurable on E,
- (iv) P is integrable on M, and
- (v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then there exists a sequence I of real numbers such that

(vi) for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$, and

(vii) if for every element x of X such that $x \in E$ holds $F\#x$ is convergent, then I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

Let X be a set and let F be a sequence of partial functions from X into $\mathbb{R}$. We say that F is uniformly bounded if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a real number K such that for every natural number n and for every element x of X if $x \in \operatorname{dom} F(0)$, then $|F(n)(x)| \leq K$.

Next we state the proposition

(49) Suppose that

(i) $M(E) < +\infty$,

(ii) $E = \operatorname{dom} F(0)$,

(iii) for every natural number n holds $F(n)$ is measurable on E,

(iv) F is uniformly bounded, and

(v) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then

(vi) for every natural number n holds $F(n)$ is integrable on M,

(vii) $\lim F$ is integrable on M, and

(viii) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

Let X be a set, let F be a sequence of partial functions from X into $\mathbb{R}$, and let f be a partial function from X to $\mathbb{R}$. We say that F is uniformly convergent to f if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) F has the same dom,

(ii) $\operatorname{dom} F(0) = \operatorname{dom} f$, and

(iii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n and for every element x of X such that $n \geq N$ and $x \in \operatorname{dom} F(0)$ holds $|F(n)(x) - f(x)| < e$.

We now state the proposition

(50) Suppose that

(i) $M(E) < +\infty$,

(ii) $E = \operatorname{dom} F(0)$,

(iii) for every natural number n holds $F(n)$ is integrable on M, and

(iv) F is uniformly convergent to f.

Then

(v) f is integrable on M, and

(vi) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int f\,\mathrm{d}M$.

In the sequel F denotes a sequence of partial functions from X into $\mathbb{C}$ with the same dom and f denotes a partial function from X to $\mathbb{C}$.

The following two propositions are true:

(51) Suppose that
(i) $E = \operatorname{dom} F(0)$,
(ii) $E = \operatorname{dom} P$,
(iii) for every natural number n holds $F(n)$ is measurable on E,
(iv) P is integrable on M,
(v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$, and
(vi) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.
Then $\lim F$ is integrable on M.

(52) Suppose that
(i) $E = \operatorname{dom} F(0)$,
(ii) $E = \operatorname{dom} P$,
(iii) for every natural number n holds $F(n)$ is measurable on E,
(iv) P is integrable on M, and
(v) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.
Then there exists a complex sequence I such that
(vi) for every natural number n holds $I(n) = \int F(n)\, \mathrm{d}M$, and
(vii) if for every element x of X such that $x \in E$ holds $F\#x$ is convergent, then I is convergent and $\lim I = \int \lim F\, \mathrm{d}M$.

Let X be a set and let F be a sequence of partial functions from X into $\mathbb{C}$. We say that F is uniformly bounded if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a real number K such that for every natural number n and for every element x of X if $x \in \operatorname{dom} F(0)$, then $|F(n)(x)| \leq K$.

The following proposition is true

(53) Suppose that
(i) $M(E) < +\infty$,
(ii) $E = \operatorname{dom} F(0)$,
(iii) for every natural number n holds $F(n)$ is measurable on E,
(iv) F is uniformly bounded, and
(v) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.
Then
(vi) for every natural number n holds $F(n)$ is integrable on M,
(vii) $\lim F$ is integrable on M, and
(viii) there exists a complex sequence I such that for every natural number n holds $I(n) = \int F(n)\, \mathrm{d}M$ and I is convergent and $\lim I = \int \lim F\, \mathrm{d}M$.

Let X be a set, let F be a sequence of partial functions from X into $\mathbb{C}$, and let f be a partial function from X to $\mathbb{C}$. We say that F is uniformly convergent to f if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) F has the same dom,

(ii) $\operatorname{dom} F(0) = \operatorname{dom} f$, and

(iii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n and for every element x of X such that $n \geq N$ and $x \in \operatorname{dom} F(0)$ holds $|F(n)(x) - f(x)| < e$.

We now state the proposition

(54) Suppose that

(i) $M(E) < +\infty$,

(ii) $E = \operatorname{dom} F(0)$,

(iii) for every natural number n holds $F(n)$ is integrable on M, and

(iv) F is uniformly convergent to f.

Then

(v) f is integrable on M, and

(vi) there exists a complex sequence I such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int f\,\mathrm{d}M$.

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