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The Cauchy-Riemann Differential Equations of Complex Functions

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Summary. In this article we prove Cauchy-Riemann differential equations of complex functions. These theorems give necessary and sufficient condition for differentiable function.

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The articles [20], [21], [6], [7], [22], [8], [3], [1], [4], [14], [13], [19], [16], [9], [2], [5], [10], [17], [11], [18], [12], and [15] provide the notation and terminology for this paper.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\Re(f)$ yielding a partial function from $\mathbb{C}$ to $\mathbb{R}$ is defined as follows:

(Def. 1) $\operatorname{dom} f = \operatorname{dom} \Re(f)$ and for every complex number z such that $z \in \operatorname{dom} \Re(f)$ holds $\Re(f)(z) = \Re(f_z)$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\Im(f)$ yields a partial function from $\mathbb{C}$ to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\operatorname{dom} f = \operatorname{dom} \Im(f)$ and for every complex number z such that $z \in \operatorname{dom} \Im(f)$ holds $\Im(f)(z) = \Im(f_z)$.

One can prove the following propositions:

(1) For every partial function f from $\mathbb{C}$ to $\mathbb{C}$ such that f is total holds $\operatorname{dom} \Re(f) = \mathbb{C}$ and $\operatorname{dom} \Im(f) = \mathbb{C}$.

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(2) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, u, v be partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, z_0 be a complex number, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose that

(i) for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} u$ and $u(\langle x, y\rangle) = \Re(f)(x + y \cdot i)$,

(ii) for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} v$ and $v(\langle x, y\rangle) = \Im(f)(x + y \cdot i)$,

(iii) $z_0 = x_0 + y_0 \cdot i$,

(iv) $x_1 = \langle x_0, y_0\rangle$, and

(v) f is differentiable in z_0.

Then

(vi) u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(vii) v is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(viii) $\Re(f'(z_0)) = \operatorname{partdiff}(u, x_1, 1)$,

(ix) $\Re(f'(z_0)) = \operatorname{partdiff}(v, x_1, 2)$,

(x) $\Im(f'(z_0)) = -\operatorname{partdiff}(u, x_1, 2)$, and

(xi) $\Im(f'(z_0)) = \operatorname{partdiff}(v, x_1, 1)$.

(3) For every sequence s of real numbers holds s is convergent and $\lim s = 0$ iff $|s|$ is convergent and $\lim|s| = 0$.

(4) Let X be a real normed space and s be a sequence of X. Then s is convergent and $\lim s = 0_X$ if and only if $\|s\|$ is convergent and $\lim\|s\| = 0$.

(5) Let u be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose $x_1 = \langle x_0, y_0\rangle$ and $\langle u\rangle$ is differentiable in x_1. Then

(i) u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(ii) $\langle \operatorname{partdiff}(u, x_1, 1)\rangle = \langle u\rangle'(x_1)(\langle 1, 0\rangle)$, and

(iii) $\langle \operatorname{partdiff}(u, x_1, 2)\rangle = \langle u\rangle'(x_1)(\langle 0, 1\rangle)$.

(6) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, u, v be partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, z_0 be a complex number, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose that for all real numbers x, y such that $\langle x, y\rangle \in \operatorname{dom} v$ holds $x+y\cdot i \in \operatorname{dom} f$ and for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} u$ and $u(\langle x, y\rangle) = \Re(f)(x + y \cdot i)$ and for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} v$ and $v(\langle x, y\rangle) = \Im(f)(x+y\cdot i)$ and $z_0 = x_0 + y_0 \cdot i$ and $x_1 = \langle x_0, y_0\rangle$ and $\langle u\rangle$ is differentiable in x_1 and $\langle v\rangle$ is differentiable in x_1 and $\operatorname{partdiff}(u, x_1, 1) = \operatorname{partdiff}(v, x_1, 2)$ and $\operatorname{partdiff}(u, x_1, 2) = -\operatorname{partdiff}(v, x_1, 1)$. Then f is differentiable in z_0 and u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2 and v is partially differentiable in

x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2 and $\Re(f'(z_0)) = \text{partdiff}(u, x_1, 1)$ and $\Re(f'(z_0)) = \text{partdiff}(v, x_1, 2)$ and $\Im(f'(z_0)) = -\text{partdiff}(u, x_1, 2)$ and $\Im(f'(z_0)) = \text{partdiff}(v, x_1, 1)$.

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