FORMALIZED MATHEMATICS
Vol. 17, No. 2, Pages 157–162, 2009
DOI: 10.2478/v10037-009-0018-6

Hopf Extension Theorem of Measure

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Summary. The authors have presented some articles about Lebesgue type integration theory. In our previous articles [12, 13, 26], we assumed that some σ-additive measure existed and that a function was measurable on that measure. However the existence of such a measure is not trivial. In general, because the construction of a finite additive measure is comparatively easy, to induce a σ-additive measure a finite additive measure is used. This is known as an E. Hopf's extension theorem of measure [15].

MML identifier: `MEASURE8`, version: `7.11.02 4.125.1059`

The articles [11], [23], [1], [24], [22], [8], [25], [10], [9], [2], [20], [26], [6], [5], [7], [13], [4], [12], [3], [16], [19], [18], [27], [21], [17], and [14] provide the notation and terminology for this paper.

1. The Outer Measure Induced by the Finite Additive Measure

For simplicity, we follow the rules: X denotes a set, F denotes a field of subsets of X, M denotes a measure on F, A, B denote subsets of X, S_1 denotes a sequence of subsets of X, s_1, s_2, s_3 denote sequences of extended reals, and n, k denote natural numbers.

We now state three propositions:

(1) $\operatorname{Ser} s_1 = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(2)[1] If s_1 is non-negative, then s_1 is summable and $\overline{\sum} s_1 = \sum s_1$.

[1]The translation of Mizar functor SUM introduced in [4] was changed from $\sum$ to $\overline{\sum}$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Suppose s_2 is non-negative and s_3 is non-negative and for every natural number n holds $s_1(n) = s_2(n)+s_3(n)$. Then s_1 is non-negative and $\overline{\sum} s_1 = \overline{\sum} s_2 + \overline{\sum} s_3$ and $\sum s_1 = \sum s_2 + \sum s_3$.

Let us consider X, F. One can check that there exists a function from $\mathbb{N}$ into F which is disjoint valued.

Let us consider X, F. A finite sequence of elements of 2^X is said to be a finite sequence of elements of F if:

(Def. 1) For every natural number k such that $k \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(k) \in F$.

Let us consider X, F. Observe that there exists a finite sequence of elements of F which is disjoint valued.

Let us consider X, F. A disjoint valued finite set sequence of F is a disjoint valued finite sequence of elements of F.

Let us consider X, F. A sequence of separated subsets of F is a disjoint valued function from $\mathbb{N}$ into F.

Let us consider X, F. A sequence of subsets of X is said to be a set sequence of F if:

(Def. 2) For every natural number n holds $\mathrm{it}(n) \in F$.

Let us consider X, A, F. A set sequence of F is said to be a covering of A in F if:

(Def. 3) $A \subseteq \bigcup \operatorname{rng} \mathrm{it}$.

In the sequel F_1 denotes a set sequence of F and C_1 denotes a covering of A in F.

Let us consider X, F, F_1, n. Then $F_1(n)$ is an element of F.

Let us consider X, F, S_1. A function from $\mathbb{N}$ into $(2^X)^{\mathbb{N}}$ is said to be a covering of S_1 in F if:

(Def. 4) For every element n of $\mathbb{N}$ holds $\mathrm{it}(n)$ is a covering of $S_1(n)$ in F.

In the sequel C_2 is a covering of S_1 in F.

Let us consider X, F, M, F_1. The functor $\mathrm{vol}(M, F_1)$ yielding a sequence of extended reals is defined as follows:

(Def. 5) For every n holds $(\mathrm{vol}(M, F_1))(n) = M(F_1(n))$.

One can prove the following proposition

(4) $\mathrm{vol}(M, F_1)$ is non-negative.

Let us consider X, F, S_1, C_2 and let n be an element of $\mathbb{N}$. Then $C_2(n)$ is a covering of $S_1(n)$ in F.

Let us consider X, F, S_1, M, C_2. The functor $\mathrm{Volume}(M, C_2)$ yielding a sequence of extended reals is defined as follows:

(Def. 6) For every element n of $\mathbb{N}$ holds $(\mathrm{Volume}(M, C_2))(n) = \overline{\sum} \mathrm{vol}(M, C_2(n))$.

The following proposition is true

(5) $0 \leq (\mathrm{Volume}(M, C_2))(n)$.

Let us consider X, F, M, A. The functor $\mathrm{Svc}(M, A)$ yielding a subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) For every extended real number x holds $x \in \mathrm{Svc}(M, A)$ iff there exists a covering C_1 of A in F such that $x = \overline{\sum} \mathrm{vol}(M, C_1)$.

Let us consider X, A, F, M. Observe that $\mathrm{Svc}(M, A)$ is non empty.

Let us consider X, F, M. The Caratheodory measure determined by M is a function from 2^X into $\overline{\mathbb{R}}$ and is defined by:

(Def. 8) For every subset A of X holds (the Caratheodory measure determined by M)$(A) = \inf \mathrm{Svc}(M, A)$.

The function InvPairFunc from $\mathbb{N}$ into $\mathbb{N} \times \mathbb{N}$ is defined by:

(Def. 9) $\mathrm{InvPairFunc} = \mathrm{PairFunc}^{-1}$.

Let us consider X, F, S_1, C_2. The functor $\mathrm{On}\, C_2$ yielding a covering of $\bigcup \mathrm{rng}\, S_1$ in F is defined by:

(Def. 10) For every natural number n holds $(\mathrm{On}\, C_2)(n) = C_2(\mathrm{pr1}(\mathrm{InvPairFunc})(n))(\mathrm{pr2}(\mathrm{InvPairFunc})(n))$.

The following propositions are true:

(6) Let k be an element of $\mathbb{N}$. Then there exists a natural number m such that for every sequence S_1 of subsets of X and for every covering C_2 of S_1 in F holds $(\sum_{\alpha=0}^{\kappa} (\mathrm{vol}(M, \mathrm{On}\, C_2))(\alpha))_{\kappa \in \mathbb{N}}(k) \leq (\sum_{\alpha=0}^{\kappa} (\mathrm{Volume}(M, C_2))(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(7) $\inf \mathrm{Svc}(M, \bigcup \mathrm{rng}\, S_1) \leq \overline{\sum} \mathrm{Volume}(M, C_2)$.

(8) If $A \in F$, then $A, \emptyset_X$ followed by $\emptyset_X$ is a covering of A in F.

(9) Let X be a set, F be a field of subsets of X, M be a measure on F, and A be a set. If $A \in F$, then (the Caratheodory measure determined by M)$(A) \leq M(A)$.

(10) The Caratheodory measure determined by M is non-negative.

(11) (The Caratheodory measure determined by M)$(\emptyset) = 0$.

(12) If $A \subseteq B$, then (the Caratheodory measure determined by M)$(A) \leq$ (the Caratheodory measure determined by M)(B).

(13) (The Caratheodory measure determined by M)$(\bigcup \mathrm{rng}\, S_1) \leq \overline{\sum}$((the Caratheodory measure determined by M) $\cdot S_1$).

(14) The Caratheodory measure determined by M is a Caratheodor's measure on X.

Let X be a set, let F be a field of subsets of X, and let M be a measure on F. Then the Caratheodory measure determined by M is a Caratheodor's measure on X.

2. Hopf Extension Theorem

Let X be a set, let F be a field of subsets of X, and let M be a measure on F. We say that M is completely-additive if and only if:

(Def. 11) For every sequence F_1 of separated subsets of F such that $\bigcup \operatorname{rng} F_1 \in F$ holds $\overline{\sum}(M \cdot F_1) = M(\bigcup \operatorname{rng} F_1)$.

The following propositions are true:

(15) The partial unions of F_1 are a set sequence of F.

(16) The partial diff-unions of F_1 are a set sequence of F.

(17) Suppose $A \in F$. Then there exists a sequence F_1 of separated subsets of F such that $A = \bigcup \operatorname{rng} F_1$ and for every natural number n holds $F_1(n) \subseteq C_1(n)$.

(18) Suppose M is completely-additive. Let A be a set. If $A \in F$, then $M(A) =$ (the Caratheodory measure determined by M)(A).

In the sequel C is a Caratheodor's measure on X.

We now state three propositions:

(19) If for every subset B of X holds $C(B \cap A) + C(B \cap (X \setminus A)) \leq C(B)$, then $A \in \sigma\text{-Field}(C)$.

(20) $F \subseteq \sigma$-Field(the Caratheodory measure determined by M).

(21) Let X be a set, F be a field of subsets of X, F_1 be a set sequence of F, and M be a function from F into $\overline{\mathbb{R}}$. Then $M \cdot F_1$ is a sequence of extended reals.

Let X be a set, let F be a field of subsets of X, let F_1 be a set sequence of F, and let g be a function from F into $\overline{\mathbb{R}}$. Then $g \cdot F_1$ is a sequence of extended reals.

One can prove the following proposition

(22) Let X be a set, S be a σ-field of subsets of X, S_2 be a sequence of subsets of S, and M be a function from S into $\overline{\mathbb{R}}$. Then $M \cdot S_2$ is a sequence of extended reals.

Let X be a set, let S be a σ-field of subsets of X, let S_2 be a sequence of subsets of S, and let g be a function from S into $\overline{\mathbb{R}}$. Then $g \cdot S_2$ is a sequence of extended reals.

Next we state several propositions:

(23) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and n be a natural number. Suppose that for every natural number m such that $m \leq n$ holds $F(m) \leq G(m)$. Then $(\operatorname{Ser} F)(n) \leq (\operatorname{Ser} G)(n)$.

(24) For all X, C and for every sequence s_1 of separated subsets of σ-Field(C) holds $\bigcup \operatorname{rng} s_1 \in \sigma\text{-Field}(C)$ and $C(\bigcup \operatorname{rng} s_1) = \sum(C \cdot s_1)$.

(25) For all X, C and for every sequence s_1 of subsets of σ-Field(C) holds $\bigcup s_1 \in \sigma\text{-Field}(C)$.

(26) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is non-decreasing, then $\lim(M \cdot S_2) = M(\lim S_2)$.

(27) If F_1 is non-decreasing, then $M \cdot F_1$ is non-decreasing.

(28) If F_1 is descending, then $M \cdot F_1$ is non-increasing.

(29) Let X be a set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is non-decreasing, then $M \cdot S_2$ is non-decreasing.

(30) Let X be a set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is descending, then $M \cdot S_2$ is non-increasing.

(31) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and S_2 be a sequence of subsets of S. If S_2 is descending and $M(S_2(0)) < +\infty$, then $\lim(M \cdot S_2) = M(\lim S_2)$.

Let X be a set, let F be a field of subsets of X, let S be a σ-field of subsets of X, let m be a measure on F, and let M be a σ-measure on S. We say that M is an extension of m if and only if:

(Def. 12) For every set A such that $A \in F$ holds $M(A) = m(A)$.

The following four propositions are true:

(32) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. If there exists a σ-measure on $\sigma(F)$ which is an extension of m, then m is completely-additive.

(33) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. Suppose m is completely-additive. Then there exists a σ-measure M on $\sigma(F)$ such that M is an extension of m and $M = \sigma$-Meas(the Caratheodory measure determined by m)$\restriction\sigma(F)$.

(34) If for every n holds $M(F_1(n)) < +\infty$, then M((the partial unions of F_1)$(k)) < +\infty$.

(35) Let X be a non empty set, F be a field of subsets of X, and m be a measure on F. Suppose that

(i) m is completely-additive, and

(ii) there exists a set sequence A_1 of F such that for every natural number n holds $m(A_1(n)) < +\infty$ and $X = \bigcup \operatorname{rng} A_1$.

Let M be a σ-measure on $\sigma(F)$. Suppose M is an extension of m. Then $M = \sigma$-Meas(the Caratheodory measure determined by m)$\restriction\sigma(F)$.

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Received April 7, 2009