

# Hopf Extension Theorem of Measure

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**Summary.** The authors have presented some articles about Lebesgue type integration theory. In our previous articles [12, 13, 26], we assumed that some  $\sigma$ -additive measure existed and that a function was measurable on that measure. However the existence of such a measure is not trivial. In general, because the construction of a finite additive measure is comparatively easy, to induce a  $\sigma$ -additive measure a finite additive measure is used. This is known as an E. Hopf's extension theorem of measure [15].

MML identifier: MEASURE8, version: 7.11.02 4.125.1059

The articles [11], [23], [1], [24], [22], [8], [25], [10], [9], [2], [20], [26], [6], [5], [7], [13], [4], [12], [3], [16], [19], [18], [27], [21], [17], and [14] provide the notation and terminology for this paper.

## 1. THE OUTER MEASURE INDUCED BY THE FINITE ADDITIVE MEASURE

For simplicity, we follow the rules:  $X$  denotes a set,  $F$  denotes a field of subsets of  $X$ ,  $M$  denotes a measure on  $F$ ,  $A, B$  denote subsets of  $X$ ,  $S_1$  denotes a sequence of subsets of  $X$ ,  $s_1, s_2, s_3$  denote sequences of extended reals, and  $n, k$  denote natural numbers.

We now state three propositions:

(1)  $\text{Ser } s_1 = (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}$ .

(2)<sup>1</sup> If  $s_1$  is non-negative, then  $s_1$  is summable and  $\overline{\sum} s_1 = \sum s_1$ .

<sup>1</sup>The translation of Mizar functor SUM introduced in [4] was changed from  $\sum$  to  $\overline{\sum}$ .

- (3) Suppose  $s_2$  is non-negative and  $s_3$  is non-negative and for every natural number  $n$  holds  $s_1(n) = s_2(n) + s_3(n)$ . Then  $s_1$  is non-negative and  $\overline{\sum} s_1 = \overline{\sum} s_2 + \overline{\sum} s_3$  and  $\sum s_1 = \sum s_2 + \sum s_3$ .

Let us consider  $X, F$ . One can check that there exists a function from  $\mathbb{N}$  into  $F$  which is disjoint valued.

Let us consider  $X, F$ . A finite sequence of elements of  $2^X$  is said to be a finite sequence of elements of  $F$  if:

- (Def. 1) For every natural number  $k$  such that  $k \in \text{dom}$  it holds  $it(k) \in F$ .

Let us consider  $X, F$ . Observe that there exists a finite sequence of elements of  $F$  which is disjoint valued.

Let us consider  $X, F$ . A disjoint valued finite set sequence of  $F$  is a disjoint valued finite sequence of elements of  $F$ .

Let us consider  $X, F$ . A sequence of separated subsets of  $F$  is a disjoint valued function from  $\mathbb{N}$  into  $F$ .

Let us consider  $X, F$ . A sequence of subsets of  $X$  is said to be a set sequence of  $F$  if:

- (Def. 2) For every natural number  $n$  holds  $it(n) \in F$ .

Let us consider  $X, A, F$ . A set sequence of  $F$  is said to be a covering of  $A$  in  $F$  if:

- (Def. 3)  $A \subseteq \bigcup \text{rng } it$ .

In the sequel  $F_1$  denotes a set sequence of  $F$  and  $C_1$  denotes a covering of  $A$  in  $F$ .

Let us consider  $X, F, F_1, n$ . Then  $F_1(n)$  is an element of  $F$ .

Let us consider  $X, F, S_1$ . A function from  $\mathbb{N}$  into  $(2^X)^\mathbb{N}$  is said to be a covering of  $S_1$  in  $F$  if:

- (Def. 4) For every element  $n$  of  $\mathbb{N}$  holds  $it(n)$  is a covering of  $S_1(n)$  in  $F$ .

In the sequel  $C_2$  is a covering of  $S_1$  in  $F$ .

Let us consider  $X, F, M, F_1$ . The functor  $\text{vol}(M, F_1)$  yielding a sequence of extended reals is defined as follows:

- (Def. 5) For every  $n$  holds  $(\text{vol}(M, F_1))(n) = M(F_1(n))$ .

One can prove the following proposition

- (4)  $\text{vol}(M, F_1)$  is non-negative.

Let us consider  $X, F, S_1, C_2$  and let  $n$  be an element of  $\mathbb{N}$ . Then  $C_2(n)$  is a covering of  $S_1(n)$  in  $F$ .

Let us consider  $X, F, S_1, M, C_2$ . The functor  $\text{Volume}(M, C_2)$  yielding a sequence of extended reals is defined as follows:

- (Def. 6) For every element  $n$  of  $\mathbb{N}$  holds  $(\text{Volume}(M, C_2))(n) = \overline{\sum} \text{vol}(M, C_2(n))$ .

The following proposition is true

- (5)  $0 \leq (\text{Volume}(M, C_2))(n)$ .

Let us consider  $X, F, M, A$ . The functor  $\text{Svc}(M, A)$  yielding a subset of  $\overline{\mathbb{R}}$  is defined as follows:

(Def. 7) For every extended real number  $x$  holds  $x \in \text{Svc}(M, A)$  iff there exists a covering  $C_1$  of  $A$  in  $F$  such that  $x = \overline{\sum} \text{vol}(M, C_1)$ .

Let us consider  $X, A, F, M$ . Observe that  $\text{Svc}(M, A)$  is non empty.

Let us consider  $X, F, M$ . The Caratheodory measure determined by  $M$  is a function from  $2^X$  into  $\overline{\mathbb{R}}$  and is defined by:

(Def. 8) For every subset  $A$  of  $X$  holds (the Caratheodory measure determined by  $M$ )( $A$ ) =  $\inf \text{Svc}(M, A)$ .

The function  $\text{InvPairFunc}$  from  $\mathbb{N}$  into  $\mathbb{N} \times \mathbb{N}$  is defined by:

(Def. 9)  $\text{InvPairFunc} = \text{PairFunc}^{-1}$ .

Let us consider  $X, F, S_1, C_2$ . The functor  $\text{On } C_2$  yielding a covering of  $\bigcup \text{rng } S_1$  in  $F$  is defined by:

(Def. 10) For every natural number  $n$  holds  $(\text{On } C_2)(n) = C_2(\text{pr1}(\text{InvPairFunc})(n))(\text{pr2}(\text{InvPairFunc})(n))$ .

The following propositions are true:

(6) Let  $k$  be an element of  $\mathbb{N}$ . Then there exists a natural number  $m$  such that for every sequence  $S_1$  of subsets of  $X$  and for every covering  $C_2$  of  $S_1$  in  $F$  holds  $(\sum_{\alpha=0}^k (\text{vol}(M, \text{On } C_2))(\alpha))_{\kappa \in \mathbb{N}}(k) \leq (\sum_{\alpha=0}^k (\text{Volume}(M, C_2))(\alpha))_{\kappa \in \mathbb{N}}(m)$ .

(7)  $\inf \text{Svc}(M, \bigcup \text{rng } S_1) \leq \overline{\sum} \text{Volume}(M, C_2)$ .

(8) If  $A \in F$ , then  $A, \emptyset_X$  followed by  $\emptyset_X$  is a covering of  $A$  in  $F$ .

(9) Let  $X$  be a set,  $F$  be a field of subsets of  $X$ ,  $M$  be a measure on  $F$ , and  $A$  be a set. If  $A \in F$ , then (the Caratheodory measure determined by  $M$ )( $A$ )  $\leq M(A)$ .

(10) The Caratheodory measure determined by  $M$  is non-negative.

(11) (The Caratheodory measure determined by  $M$ )( $\emptyset$ ) = 0.

(12) If  $A \subseteq B$ , then (the Caratheodory measure determined by  $M$ )( $A$ )  $\leq$  (the Caratheodory measure determined by  $M$ )( $B$ ).

(13) (The Caratheodory measure determined by  $M$ )( $\bigcup \text{rng } S_1$ )  $\leq \overline{\sum} ((\text{the Caratheodory measure determined by } M) \cdot S_1)$ .

(14) The Caratheodory measure determined by  $M$  is a Caratheodor's measure on  $X$ .

Let  $X$  be a set, let  $F$  be a field of subsets of  $X$ , and let  $M$  be a measure on  $F$ . Then the Caratheodory measure determined by  $M$  is a Caratheodor's measure on  $X$ .

## 2. HOPF EXTENSION THEOREM

Let  $X$  be a set, let  $F$  be a field of subsets of  $X$ , and let  $M$  be a measure on  $F$ . We say that  $M$  is completely-additive if and only if:

(Def. 11) For every sequence  $F_1$  of separated subsets of  $F$  such that  $\bigcup \text{rng } F_1 \in F$  holds  $\overline{\sum}(M \cdot F_1) = M(\bigcup \text{rng } F_1)$ .

The following propositions are true:

- (15) The partial unions of  $F_1$  are a set sequence of  $F$ .
- (16) The partial diff-unions of  $F_1$  are a set sequence of  $F$ .
- (17) Suppose  $A \in F$ . Then there exists a sequence  $F_1$  of separated subsets of  $F$  such that  $A = \bigcup \text{rng } F_1$  and for every natural number  $n$  holds  $F_1(n) \subseteq C_1(n)$ .
- (18) Suppose  $M$  is completely-additive. Let  $A$  be a set. If  $A \in F$ , then  $M(A) =$  (the Caratheodory measure determined by  $M$ )( $A$ ).

In the sequel  $C$  is a Caratheodor's measure on  $X$ .

We now state three propositions:

- (19) If for every subset  $B$  of  $X$  holds  $C(B \cap A) + C(B \cap (X \setminus A)) \leq C(B)$ , then  $A \in \sigma\text{-Field}(C)$ .
- (20)  $F \subseteq \sigma\text{-Field}$ (the Caratheodory measure determined by  $M$ ).
- (21) Let  $X$  be a set,  $F$  be a field of subsets of  $X$ ,  $F_1$  be a set sequence of  $F$ , and  $M$  be a function from  $F$  into  $\overline{\mathbb{R}}$ . Then  $M \cdot F_1$  is a sequence of extended reals.

Let  $X$  be a set, let  $F$  be a field of subsets of  $X$ , let  $F_1$  be a set sequence of  $F$ , and let  $g$  be a function from  $F$  into  $\overline{\mathbb{R}}$ . Then  $g \cdot F_1$  is a sequence of extended reals.

One can prove the following proposition

- (22) Let  $X$  be a set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $S_2$  be a sequence of subsets of  $S$ , and  $M$  be a function from  $S$  into  $\overline{\mathbb{R}}$ . Then  $M \cdot S_2$  is a sequence of extended reals.

Let  $X$  be a set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $S_2$  be a sequence of subsets of  $S$ , and let  $g$  be a function from  $S$  into  $\overline{\mathbb{R}}$ . Then  $g \cdot S_2$  is a sequence of extended reals.

Next we state several propositions:

- (23) Let  $F, G$  be functions from  $\mathbb{N}$  into  $\overline{\mathbb{R}}$  and  $n$  be a natural number. Suppose that for every natural number  $m$  such that  $m \leq n$  holds  $F(m) \leq G(m)$ . Then  $(\text{Ser } F)(n) \leq (\text{Ser } G)(n)$ .
- (24) For all  $X, C$  and for every sequence  $s_1$  of separated subsets of  $\sigma\text{-Field}(C)$  holds  $\bigcup \text{rng } s_1 \in \sigma\text{-Field}(C)$  and  $C(\bigcup \text{rng } s_1) = \sum(C \cdot s_1)$ .
- (25) For all  $X, C$  and for every sequence  $s_1$  of subsets of  $\sigma\text{-Field}(C)$  holds  $\bigcup s_1 \in \sigma\text{-Field}(C)$ .

- (26) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $S_2$  be a sequence of subsets of  $S$ . If  $S_2$  is non-decreasing, then  $\lim(M \cdot S_2) = M(\lim S_2)$ .
- (27) If  $F_1$  is non-decreasing, then  $M \cdot F_1$  is non-decreasing.
- (28) If  $F_1$  is descending, then  $M \cdot F_1$  is non-increasing.
- (29) Let  $X$  be a set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $S_2$  be a sequence of subsets of  $S$ . If  $S_2$  is non-decreasing, then  $M \cdot S_2$  is non-decreasing.
- (30) Let  $X$  be a set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $S_2$  be a sequence of subsets of  $S$ . If  $S_2$  is descending, then  $M \cdot S_2$  is non-increasing.
- (31) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $S_2$  be a sequence of subsets of  $S$ . If  $S_2$  is descending and  $M(S_2(0)) < +\infty$ , then  $\lim(M \cdot S_2) = M(\lim S_2)$ .

Let  $X$  be a set, let  $F$  be a field of subsets of  $X$ , let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $m$  be a measure on  $F$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . We say that  $M$  is an extension of  $m$  if and only if:

(Def. 12) For every set  $A$  such that  $A \in F$  holds  $M(A) = m(A)$ .

The following four propositions are true:

- (32) Let  $X$  be a non empty set,  $F$  be a field of subsets of  $X$ , and  $m$  be a measure on  $F$ . If there exists a  $\sigma$ -measure on  $\sigma(F)$  which is an extension of  $m$ , then  $m$  is completely-additive.
- (33) Let  $X$  be a non empty set,  $F$  be a field of subsets of  $X$ , and  $m$  be a measure on  $F$ . Suppose  $m$  is completely-additive. Then there exists a  $\sigma$ -measure  $M$  on  $\sigma(F)$  such that  $M$  is an extension of  $m$  and  $M = \sigma\text{-Meas}(\text{the Caratheodory measure determined by } m) \upharpoonright \sigma(F)$ .
- (34) If for every  $n$  holds  $M(F_1(n)) < +\infty$ , then  $M(\text{the partial unions of } F_1)(k) < +\infty$ .
- (35) Let  $X$  be a non empty set,  $F$  be a field of subsets of  $X$ , and  $m$  be a measure on  $F$ . Suppose that
  - (i)  $m$  is completely-additive, and
  - (ii) there exists a set sequence  $A_1$  of  $F$  such that for every natural number  $n$  holds  $m(A_1(n)) < +\infty$  and  $X = \bigcup \text{rng } A_1$ .
 Let  $M$  be a  $\sigma$ -measure on  $\sigma(F)$ . Suppose  $M$  is an extension of  $m$ . Then  $M = \sigma\text{-Meas}(\text{the Caratheodory measure determined by } m) \upharpoonright \sigma(F)$ .

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*Received April 7, 2009*

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