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# Labelled State Transition Systems

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**Summary.** This article introduces labelled state transition systems, where transitions may be labelled by words from a given alphabet. Reduction relations from [4] are used to define transitions between states, acceptance of words, and reachable states. Deterministic transition systems are also defined.

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The articles [1], [8], [2], [11], [6], [17], [7], [9], [16], [15], [14], [4], [10], [13], [3], [12], and [5] provide the notation and terminology for this paper.

## 1. Preliminaries

For simplicity, we adopt the following convention: $x$, $x_1$, $x_2$, $y$, $y_1$, $y_2$, $z$, $z_1$, $z_2$, $X$, $X_1$, $X_2$ are sets, $E$ is a non empty set, $e$ is an element of $E$, $u$, $v$, $v_1$, $v_2$, $w$, $w_1$, $w_2$ are elements of $E^\omega$, $F$, $F_1$, $F_2$ are subsets of $E^\omega$, and $k$, $l$ are natural numbers.

Next we state a number of propositions:

(1) For every finite sequence $p$ such that $k \in \operatorname{dom} p$ holds $(\langle x \rangle \frown p)(k+1) = p(k)$.

(2) For every finite sequence $p$ such that $p \neq \emptyset$ there exists a finite sequence $q$ and there exists $x$ such that $p = q \frown \langle x \rangle$ and $\operatorname{len} p = \operatorname{len} q + 1$.

(3) For every finite sequence $p$ such that $k \in \operatorname{dom} p$ and $k+1 \notin \operatorname{dom} p$ holds $\operatorname{len} p = k$.

(4) Let $R$ be a binary relation, $P$ be a reduction sequence w.r.t. $R$, and $q_1$, $q_2$ be finite sequences. Suppose $P = q_1 \frown q_2$ and $\operatorname{len} q_1 > 0$ and $\operatorname{len} q_2 > 0$. Then $q_1$ is a reduction sequence w.r.t. $R$ and $q_2$ is a reduction sequence w.r.t. $R$.




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(5) Let $R$ be a binary relation and $P$ be a reduction sequence w.r.t. $R$. Suppose $\operatorname{len} P > 1$. Then there exists a reduction sequence $Q$ w.r.t. $R$ such that $\langle P(1)\rangle \frown Q = P$ and $\operatorname{len} Q + 1 = \operatorname{len} P$.

(6) Let $R$ be a binary relation and $P$ be a reduction sequence w.r.t. $R$. Suppose $\operatorname{len} P > 1$. Then there exists a reduction sequence $Q$ w.r.t. $R$ such that $Q \frown \langle P(\operatorname{len} P)\rangle = P$ and $\operatorname{len} Q + 1 = \operatorname{len} P$.

(7) Let $R$ be a binary relation and $P$ be a reduction sequence w.r.t. $R$. Suppose $\operatorname{len} P > 1$. Then there exists a reduction sequence $Q$ w.r.t. $R$ such that $\operatorname{len} Q + 1 = \operatorname{len} P$ and for every $k$ such that $k \in \operatorname{dom} Q$ holds $Q(k) = P(k+1)$.

(8) For every binary relation $R$ such that $\langle x, y\rangle$ is a reduction sequence w.r.t. $R$ holds $\langle x, y\rangle \in R$.

(9) If $w = u \frown v$, then $\operatorname{len} u \leq \operatorname{len} w$ and $\operatorname{len} v \leq \operatorname{len} w$.

(10) If $w = u \frown v$ and $u \neq \langle\rangle_E$ and $v \neq \langle\rangle_E$, then $\operatorname{len} u < \operatorname{len} w$ and $\operatorname{len} v < \operatorname{len} w$.

(11) If $w_1 \frown v_1 = w_2 \frown v_2$ and if $\operatorname{len} w_1 = \operatorname{len} w_2$ or $\operatorname{len} v_1 = \operatorname{len} v_2$, then $w_1 = w_2$ and $v_1 = v_2$.

(12) If $w_1 \frown v_1 = w_2 \frown v_2$ and if $\operatorname{len} w_1 \leq \operatorname{len} w_2$ or $\operatorname{len} v_1 \geq \operatorname{len} v_2$, then there exists $u$ such that $w_1 \frown u = w_2$ and $v_1 = u \frown v_2$.

(13) If $w_1 \frown v_1 = w_2 \frown v_2$, then there exists $u$ such that $w_1 \frown u = w_2$ and $v_1 = u \frown v_2$ or there exists $u$ such that $w_2 \frown u = w_1$ and $v_2 = u \frown v_1$.

Let us consider $X$. We introduce transition-systems over $X$ which are extensions of 1-sorted structure and are systems

$\langle$ a carrier, a transition $\rangle$,

where the carrier is a set and the transition is a relation between the carrier$\times$ $X$ and the carrier.

## 2. Transition Systems over Subsets of $E^\omega$

Let us consider $E$, $F$ and let $\mathfrak{T}$ be a transition-system over $F$. We say that $\mathfrak{T}$ is deterministic if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) The transition of $\mathfrak{T}$ is a function,

(ii) $\langle\rangle_E \notin \operatorname{rng} \operatorname{dom}$ (the transition of $\mathfrak{T}$), and

(iii) for every element $s$ of $\mathfrak{T}$ and for all $u$, $v$ such that $u \neq v$ and $\langle s, u\rangle \in \operatorname{dom}$ (the transition of $\mathfrak{T}$) and $\langle s, v\rangle \in \operatorname{dom}$ (the transition of $\mathfrak{T}$) it is not true that there exists $w$ such that $u \frown w = v$ or $v \frown w = u$.

We now state the proposition

(14) For every transition-system $\mathfrak{T}$ over $F$ such that $\operatorname{dom}$ (the transition of $\mathfrak{T}$) $= \emptyset$ holds $\mathfrak{T}$ is deterministic.

Let us consider $E$, $F$. Observe that there exists a transition-system over $F$ which is strict, non empty, finite, and deterministic.

## 3. Productions

Let us consider $X$, let $\mathfrak{T}$ be a transition-system over $X$, and let us consider $x$, $y$, $z$. The predicate $x, y \to_{\mathfrak{T}} z$ is defined by:

(Def. 2) $\langle\langle x, y\rangle, z\rangle \in$ the transition of $\mathfrak{T}$.

We now state several propositions:

(15) Let $\mathfrak{T}$ be a transition-system over $X$. Suppose $x, y \to_{\mathfrak{T}} z$. Then

- (i) $x \in \mathfrak{T}$,
- (ii) $y \in X$,
- (iii) $z \in \mathfrak{T}$,
- (iv) $x \in \operatorname{dom}\operatorname{dom}$ (the transition of $\mathfrak{T}$),
- (v) $y \in \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), and
- (vi) $z \in \operatorname{rng}$ (the transition of $\mathfrak{T}$).

(16) Let $\mathfrak{T}_1$ be a transition-system over $X_1$ and $\mathfrak{T}_2$ be a transition-system over $X_2$. Suppose the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. If $x, y \to_{\mathfrak{T}_1} z$, then $x, y \to_{\mathfrak{T}_2} z$.

(17) Let $\mathfrak{T}$ be a transition-system over $F$. Suppose the transition of $\mathfrak{T}$ is a function. If $x, y \to_{\mathfrak{T}} z_1$ and $x, y \to_{\mathfrak{T}} z_2$, then $z_1 = z_2$.

(18) For every deterministic transition-system $\mathfrak{T}$ over $F$ such that $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$) holds $x, \langle\rangle_E \not\to_{\mathfrak{T}} y$.

(19) Let $\mathfrak{T}$ be a deterministic transition-system over $F$. If $u \neq v$ and $x, u \to_{\mathfrak{T}} z_1$ and $x, v \to_{\mathfrak{T}} z_2$, then it is not true that there exists $w$ such that $u ^\frown w = v$ or $v ^\frown w = u$.

## 4. Direct Transitions

Let us consider $E$, $F$, let $\mathfrak{T}$ be a transition-system over $F$, and let us consider $x_1$, $x_2$, $y_1$, $y_2$. The predicate $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ is defined as follows:

(Def. 3) There exist $v$, $w$ such that $v = y_2$ and $x_1, w \to_{\mathfrak{T}} y_1$ and $x_2 = w ^\frown v$.

The following propositions are true:

(20) Let $\mathfrak{T}$ be a transition-system over $F$. Suppose $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$. Then $x_1$, $y_1 \in \mathfrak{T}$ and $x_2$, $y_2 \in E^\omega$ and $x_1 \in \operatorname{dom}\operatorname{dom}$ (the transition of $\mathfrak{T}$) and $y_1 \in \operatorname{rng}$ (the transition of $\mathfrak{T}$).

(21) Let $\mathfrak{T}_1$ be a transition-system over $F_1$ and $\mathfrak{T}_2$ be a transition-system over $F_2$. Suppose the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}_1} y_1, y_2$. Then $x_1, x_2 \Rightarrow_{\mathfrak{T}_2} y_1, y_2$.

(22) For every transition-system $\mathfrak{T}$ over $F$ such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ there exists $w$ such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \frown v$.

(23) For every transition-system $\mathfrak{T}$ over $F$ holds $x, y \rightarrow_{\mathfrak{T}} z$ iff $x, y \Rightarrow_{\mathfrak{T}} z, \langle\rangle_E$.

(24) For every transition-system $\mathfrak{T}$ over $F$ holds $x, v \rightarrow_{\mathfrak{T}} y$ iff $x, v \frown w \Rightarrow_{\mathfrak{T}} y, w$.

(25) For every transition-system $\mathfrak{T}$ over $F$ such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ holds $x, u \frown w \Rightarrow_{\mathfrak{T}} y, v \frown w$.

(26) For every transition-system $\mathfrak{T}$ over $F$ such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ holds $\operatorname{len} u \geq \operatorname{len} v$.

(27) Let $\mathfrak{T}$ be a transition-system over $F$. Suppose the transition of $\mathfrak{T}$ is a function. If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z$, then $y_1 = y_2$.

(28) For every transition-system $\mathfrak{T}$ over $F$ such that $\langle\rangle_E \notin \operatorname{rng} \operatorname{dom}$ (the transition of $\mathfrak{T}$) holds $x, z \not\Rightarrow_{\mathfrak{T}} y, z$.

(29) For every transition-system $\mathfrak{T}$ over $F$ such that $\langle\rangle_E \notin \operatorname{rng} \operatorname{dom}$ (the transition of $\mathfrak{T}$) holds if $x, u \Rightarrow_{\mathfrak{T}} y, v$, then $\operatorname{len} u > \operatorname{len} v$.

(30) For every deterministic transition-system $\mathfrak{T}$ over $F$ such that $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z_1$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z_2$ holds $y_1 = y_2$ and $z_1 = z_2$.

## 5. Reduction Relation

In the sequel $\mathfrak{T}$ is a non empty transition-system over $F$, $s$, $t$ are elements of $\mathfrak{T}$, and $S$ is a subset of $\mathfrak{T}$.

Let us consider $E$, $F$, $\mathfrak{T}$. The functor $\Rightarrow_{\mathfrak{T}}$ yielding a binary relation on (the carrier of $\mathfrak{T}$) $\times E^{\omega}$ is defined as follows:

(Def. 4) $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$ iff $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$.

The following propositions are true:

(31) If $\langle x, y\rangle \in \Rightarrow_{\mathfrak{T}}$, then there exist $s$, $v$, $t$, $w$ such that $x = \langle s, v\rangle$ and $y = \langle t, w\rangle$.

(32) Suppose $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$. Then $x_1$, $y_1 \in \mathfrak{T}$ and $x_2$, $y_2 \in E^{\omega}$ and $x_1 \in \operatorname{dom} \operatorname{dom}$ (the transition of $\mathfrak{T}$) and $y_1 \in \operatorname{rng}$ (the transition of $\mathfrak{T}$).

(33) If $x \in \Rightarrow_{\mathfrak{T}}$, then there exist $s$, $t$, $v$, $w$ such that $x = \langle\langle s, v\rangle, \langle t, w\rangle\rangle$.

(34) Let $\mathfrak{T}_1$ be a non empty transition-system over $F_1$ and $\mathfrak{T}_2$ be a non empty transition-system over $F_2$. Suppose the carrier of $\mathfrak{T}_1$ = the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1$ = the transition of $\mathfrak{T}_2$. Then $\Rightarrow_{\mathfrak{T}_1} = \Rightarrow_{\mathfrak{T}_2}$.

(35) If $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then there exist $v$, $w$ such that $v = y_2$ and $x_1, w \rightarrow_{\mathfrak{T}} y_1$ and $x_2 = w \frown v$.

(36) If $\langle\langle x, u\rangle, \langle y, v\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then there exists $w$ such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \frown v$.

(37) $x, y \rightarrow_{\mathfrak{T}} z$ iff $\langle\langle x, y\rangle, \langle z, \langle\rangle_E\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$.

(38) $x, v \rightarrow_{\mathfrak{T}} y$ iff $\langle\langle x, v \frown w\rangle, \langle y, w\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$.

(39) If $\langle\langle x, u\rangle, \langle y, v\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then $\langle\langle x, u \frown w\rangle, \langle y, v \frown w\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$.

(40) If $\langle\langle x, u\rangle, \langle y, v\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then $\operatorname{len} u \geq \operatorname{len} v$.

(41) If the transition of $\mathfrak{T}$ is a function, then if $\langle x, \langle y_1, z\rangle\rangle, \langle x, \langle y_2, z\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$.

(42) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $\langle\langle x, u\rangle, \langle y, v\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then $\operatorname{len} u > \operatorname{len} v$.

(43) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then $\langle\langle x, z\rangle, \langle y, z\rangle\rangle \notin \Rightarrow_{\mathfrak{T}}$.

(44) If $\mathfrak{T}$ is deterministic, then if $\langle x, y_1\rangle, \langle x, y_2\rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$.

(45) If $\mathfrak{T}$ is deterministic, then if $\langle x, \langle y_1, z_1\rangle\rangle, \langle x, \langle y_2, z_2\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$ and $z_1 = z_2$.

(46) If $\mathfrak{T}$ is deterministic, then $\Rightarrow_{\mathfrak{T}}$ is function-like.

## 6. Reduction Sequences

Let us consider $x$, $E$. The functor $\dim_2(x, E)$ yields an element of $E^\omega$ and is defined as follows:

(Def. 5) $$\dim_2(x, E) = \begin{cases} x_{\mathbf{2}}, & \text{if there exist } y, u \text{ such that } x = \langle y, u\rangle, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state a number of propositions:

(47) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. If $k, k+1 \in \operatorname{dom} P$, then there exist $s$, $v$, $t$, $w$ such that $P(k) = \langle s, v\rangle$ and $P(k+1) = \langle t, w\rangle$.

(48) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. If $k, k+1 \in \operatorname{dom} P$, then $P(k) = \langle P(k)_{\mathbf{1}}, P(k)_{\mathbf{2}}\rangle$ and $P(k+1) = \langle P(k+1)_{\mathbf{1}}, P(k+1)_{\mathbf{2}}\rangle$.

(49) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. Suppose $k, k+1 \in \operatorname{dom} P$. Then

- (i) $P(k)_{\mathbf{1}} \in \mathfrak{T}$,
- (ii) $P(k)_{\mathbf{2}} \in E^\omega$,
- (iii) $P(k+1)_{\mathbf{1}} \in \mathfrak{T}$,
- (iv) $P(k+1)_{\mathbf{2}} \in E^\omega$,
- (v) $P(k)_{\mathbf{1}} \in \operatorname{dom}\operatorname{dom}$ (the transition of $\mathfrak{T}$), and
- (vi) $P(k+1)_{\mathbf{1}} \in \operatorname{rng}$ (the transition of $\mathfrak{T}$).

(50) Let $\mathfrak{T}_1$ be a non empty transition-system over $F_1$ and $\mathfrak{T}_2$ be a non empty transition-system over $F_2$. Suppose the carrier of $\mathfrak{T}_1$ = the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1$ = the transition of $\mathfrak{T}_2$. Then every reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}_1}$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}_2}$.

(51) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If there exist $x$, $u$ such that $P(1) = \langle x, u\rangle$, then for every $k$ such that $k \in \operatorname{dom} P$ holds $\dim_2(P(k), E) = P(k)_{\mathbf{2}}$.

(52) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(\operatorname{len} P) = \langle y, w\rangle$, then for every $k$ such that $k \in \operatorname{dom} P$ there exists $u$ such that $P(k)_{\mathbf{2}} = u \frown w$.

(53) For every reduction sequence $P$ w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = \langle x, v\rangle$ and $P(\operatorname{len} P) = \langle y, w\rangle$ there exists $u$ such that $v = u \frown w$.

(54) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u\rangle$ and $P(\operatorname{len} P) = \langle y, u\rangle$, then for every $k$ such that $k \in \operatorname{dom} P$ holds $P(k)_{\mathbf{2}} = u$.

(55) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. Suppose $k$, $k+1 \in \operatorname{dom} P$. Then there exist $v$, $w$ such that $v = P(k+1)_{\mathbf{2}}$ and $P(k)_{\mathbf{1}}, w \rightarrow_{\mathfrak{T}} P(k+1)_{\mathbf{1}}$ and $P(k)_{\mathbf{2}} = w \frown v$.

(56) Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. Suppose $k$, $k+1 \in \operatorname{dom} P$ and $P(k) = \langle x, u\rangle$ and $P(k+1) = \langle y, v\rangle$. Then there exists $w$ such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \frown v$.

(57) $x, y \rightarrow_{\mathfrak{T}} z$ iff $\langle\langle x, y\rangle, \langle z, \langle\rangle_E\rangle\rangle$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$.

(58) $x, v \rightarrow_{\mathfrak{T}} y$ iff $\langle\langle x, v \frown w\rangle, \langle y, w\rangle\rangle$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$.

(59) For every reduction sequence $P$ w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = \langle x, v\rangle$ and $P(\operatorname{len} P) = \langle y, w\rangle$ holds $\operatorname{len} v \geq \operatorname{len} w$.

(60) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u\rangle$ and $P(\operatorname{len} P) = \langle y, u\rangle$, then $\operatorname{len} P = 1$ and $x = y$.

(61) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1)_{\mathbf{2}} = P(\operatorname{len} P)_{\mathbf{2}}$, then $\operatorname{len} P = 1$.

(62) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u\rangle$ and $P(\operatorname{len} P) = \langle y, \langle\rangle_E\rangle$, then $\operatorname{len} P \leq \operatorname{len} u + 1$.

(63) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, \langle e\rangle\rangle$ and $P(\operatorname{len} P) = \langle y, \langle\rangle_E\rangle$, then $\operatorname{len} P = 2$.

(64) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, v\rangle$ and $P(\operatorname{len} P) = \langle y, w\rangle$, then $\operatorname{len} v > \operatorname{len} w$ or $\operatorname{len} P = 1$ and $x = y$ and $v = w$.

(65) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$. If $k$, $k+1 \in \operatorname{dom} P$, then $P(k)_{\mathbf{2}} \neq P(k+1)_{\mathbf{2}}$.

(66) Suppose $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$). Let $P$ be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given $k$, $l$. If $k$, $l \in \operatorname{dom} P$ and $k < l$, then $P(k)_{\mathbf{2}} \neq P(l)_{\mathbf{2}}$.

(67) Suppose $\mathfrak{T}$ is deterministic. Let $P$, $Q$ be reduction sequences w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = Q(1)$, then for every $k$ such that $k \in \operatorname{dom} P$ and $k \in \operatorname{dom} Q$ holds $P(k) = Q(k)$.

(68) If $\mathfrak{T}$ is deterministic, then for all reduction sequences $P$, $Q$ w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = Q(1)$ and $\operatorname{len} P = \operatorname{len} Q$ holds $P = Q$.

(69) Suppose $\mathfrak{T}$ is deterministic. Let $P$, $Q$ be reduction sequences w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = Q(1)$ and $P(\operatorname{len} P)_{\mathbf{2}} = Q(\operatorname{len} Q)_{\mathbf{2}}$, then $P = Q$.

## 7. Reductions

The following propositions are true:

(70) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v\rangle$ to $\langle y, w\rangle$, then there exists $u$ such that $v = u \frown w$.

(71) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u\rangle$ to $\langle y, v\rangle$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \frown w\rangle$ to $\langle y, v \frown w\rangle$.

(72) If $x, y \rightarrow_{\mathfrak{T}} z$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, y\rangle$ to $\langle z, \langle\rangle_E\rangle$.

(73) If $x, v \rightarrow_{\mathfrak{T}} y$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v \frown w\rangle$ to $\langle y, w\rangle$.

(74) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x_1, x_2\rangle$ to $\langle y_1, y_2\rangle$.

(75) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v\rangle$ to $\langle y, w\rangle$, then $\operatorname{len} v \geq \operatorname{len} w$.

(76) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, w\rangle$ to $\langle y, v \frown w\rangle$, then $v = \langle\rangle_E$.

(77) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v\rangle$ to $\langle y, w\rangle$, then $\operatorname{len} v > \operatorname{len} w$ or $x = y$ and $v = w$.

(78) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u\rangle$ to $\langle y, u\rangle$, then $x = y$.

(79) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, \langle e\rangle\rangle$ to $\langle y, \langle\rangle_E\rangle$, then $\langle\langle x, \langle e\rangle\rangle, \langle y, \langle\rangle_E\rangle\rangle \in \Rightarrow_{\mathfrak{T}}$.

(80) If $\mathfrak{T}$ is deterministic, then if $\Rightarrow_{\mathfrak{T}}$ reduces $x$ to $\langle y_1, z\rangle$ and $\Rightarrow_{\mathfrak{T}}$ reduces $x$ to $\langle y_2, z\rangle$, then $y_1 = y_2$.

## 8. Transitions

Let us consider $E$, $F$, $\mathfrak{T}$, $x_1$, $x_2$, $y_1$, $y_2$. The predicate $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_1, y_2$ is defined as follows:

(Def. 6) $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x_1, x_2\rangle$ to $\langle y_1, y_2\rangle$.

We now state a number of propositions:

(81) Let $\mathfrak{T}_1$ be a non empty transition-system over $F_1$ and $\mathfrak{T}_2$ be a non empty transition-system over $F_2$. Suppose the carrier of $\mathfrak{T}_1$ = the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1$ = the transition of $\mathfrak{T}_2$. If $x_1, x_2 \Rightarrow^*_{\mathfrak{T}_1} y_1, y_2$, then $x_1, x_2 \Rightarrow^*_{\mathfrak{T}_2} y_1, y_2$.

(82) $x, y \Rightarrow^*_{\mathfrak{T}} x, y$.

(83) If $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_1, y_2$ and $y_1, y_2 \Rightarrow^*_{\mathfrak{T}} z_1, z_2$, then $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} z_1, z_2$.

(84) If $x, y \rightarrow_{\mathfrak{T}} z$, then $x, y \Rightarrow^*_{\mathfrak{T}} z, \langle\rangle_E$.

(85) If $x, v \rightarrow_{\mathfrak{T}} y$, then $x, v \frown w \Rightarrow^*_{\mathfrak{T}} y, w$.

(86) If $x, u \Rightarrow^*_{\mathfrak{T}} y, v$, then $x, u \frown w \Rightarrow^*_{\mathfrak{T}} y, v \frown w$.

(87) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$, then $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_1, y_2$.

(88) If $x, v \Rightarrow^*_{\mathfrak{T}} y, w$, then there exists $u$ such that $v = u \frown w$.

(89) If $x, v \Rightarrow^*_{\mathfrak{T}} y, w$, then $\operatorname{len} w \leq \operatorname{len} v$.

(90) If $x, w \Rightarrow^*_{\mathfrak{T}} y, v \frown w$, then $v = \langle\rangle_E$.

(91) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then $x, u \Rightarrow^*_{\mathfrak{T}} y, u$ iff $x = y$.

(92) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $x, \langle e\rangle \Rightarrow^*_{\mathfrak{T}} y, \langle\rangle_E$, then $x, \langle e\rangle \Rightarrow_{\mathfrak{T}} y, \langle\rangle_E$.

(93) If $\mathfrak{T}$ is deterministic, then if $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_1, z$ and $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_2, z$, then $y_1 = y_2$.

## 9. Acceptance of Words

Let us consider $E$, $F$, $\mathfrak{T}$, $x_1$, $x_2$, $y$. The predicate $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y$ is defined as follows:

(Def. 7) $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y, \langle\rangle_E$.

We now state several propositions:

(94) Let $\mathfrak{T}_1$ be a non empty transition-system over $F_1$ and $\mathfrak{T}_2$ be a non empty transition-system over $F_2$. Suppose the carrier of $\mathfrak{T}_1$ = the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1$ = the transition of $\mathfrak{T}_2$. If $x, y \Rightarrow^*_{\mathfrak{T}_1} z$, then $x, y \Rightarrow^*_{\mathfrak{T}_2} z$.

(95) $x, \langle\rangle_E \Rightarrow^*_{\mathfrak{T}} x$.

(96) If $x, u \Rightarrow^*_{\mathfrak{T}} y$, then $x, u \frown v \Rightarrow^*_{\mathfrak{T}} y, v$.

(97) If $x, y \rightarrow_{\mathfrak{T}} z$, then $x, y \Rightarrow^*_{\mathfrak{T}} z$.

(98) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y, \langle\rangle_E$, then $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y$.

(99) If $x, u \Rightarrow^*_{\mathfrak{T}} y$ and $y, v \Rightarrow^*_{\mathfrak{T}} z$, then $x, u \frown v \Rightarrow^*_{\mathfrak{T}} z$.

(100) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then $x, \langle\rangle_E \Rightarrow^*_{\mathfrak{T}} y$ iff $x = y$.

(101) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then if $x, \langle e\rangle \Rightarrow^*_{\mathfrak{T}} y$, then $x, \langle e\rangle \Rightarrow_{\mathfrak{T}} y, \langle\rangle_E$.

(102) If $\mathfrak{T}$ is deterministic, then if $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_1$ and $x_1, x_2 \Rightarrow^*_{\mathfrak{T}} y_2$, then $y_1 = y_2$.

## 10. Reachable States

Let us consider $E$, $F$, $\mathfrak{T}$, $x$, $X$. The functor $x\text{-succ}_{\mathfrak{T}}(X)$ yields a subset of $\mathfrak{T}$ and is defined as follows:

(Def. 8) $x\text{-succ}_{\mathfrak{T}}(X) = \{s : \bigvee_t (t \in X \ \wedge\ t, x \Rightarrow^*_{\mathfrak{T}} s)\}$.

The following propositions are true:

(103) $s \in x\text{-succ}_{\mathfrak{T}}(X)$ iff there exists $t$ such that $t \in X$ and $t, x \Rightarrow^*_{\mathfrak{T}} s$.

(104) If $\langle\rangle_E \notin \operatorname{rng}\operatorname{dom}$ (the transition of $\mathfrak{T}$), then $\langle\rangle_E\text{-succ}_{\mathfrak{T}}(S) = S$.

(105) Let $\mathfrak{T}_1$ be a non empty transition-system over $F_1$ and $\mathfrak{T}_2$ be a non empty transition-system over $F_2$. Suppose the carrier of $\mathfrak{T}_1$ = the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1$ = the transition of $\mathfrak{T}_2$. Then $x\text{-succ}_{\mathfrak{T}_1}(X) = x\text{-succ}_{\mathfrak{T}_2}(X)$.

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