

# Labelled State Transition Systems

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**Summary.** This article introduces labelled state transition systems, where transitions may be labelled by words from a given alphabet. Reduction relations from [4] are used to define transitions between states, acceptance of words, and reachable states. Deterministic transition systems are also defined.

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The articles [1], [8], [2], [11], [6], [17], [7], [9], [16], [15], [14], [4], [10], [13], [3], [12], and [5] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

For simplicity, we adopt the following convention:  $x, x_1, x_2, y, y_1, y_2, z, z_1, z_2, X, X_1, X_2$  are sets,  $E$  is a non empty set,  $e$  is an element of  $E$ ,  $u, v, v_1, v_2, w, w_1, w_2$  are elements of  $E^\omega$ ,  $F, F_1, F_2$  are subsets of  $E^\omega$ , and  $k, l$  are natural numbers.

Next we state a number of propositions:

- (1) For every finite sequence  $p$  such that  $k \in \text{dom } p$  holds  $(\langle x \rangle \frown p)(k+1) = p(k)$ .
- (2) For every finite sequence  $p$  such that  $p \neq \emptyset$  there exists a finite sequence  $q$  and there exists  $x$  such that  $p = q \frown \langle x \rangle$  and  $\text{len } p = \text{len } q + 1$ .
- (3) For every finite sequence  $p$  such that  $k \in \text{dom } p$  and  $k+1 \notin \text{dom } p$  holds  $\text{len } p = k$ .
- (4) Let  $R$  be a binary relation,  $P$  be a reduction sequence w.r.t.  $R$ , and  $q_1, q_2$  be finite sequences. Suppose  $P = q_1 \frown q_2$  and  $\text{len } q_1 > 0$  and  $\text{len } q_2 > 0$ . Then  $q_1$  is a reduction sequence w.r.t.  $R$  and  $q_2$  is a reduction sequence w.r.t.  $R$ .

- (5) Let  $R$  be a binary relation and  $P$  be a reduction sequence w.r.t.  $R$ . Suppose  $\text{len } P > 1$ . Then there exists a reduction sequence  $Q$  w.r.t.  $R$  such that  $\langle P(1) \rangle \hat{\ } Q = P$  and  $\text{len } Q + 1 = \text{len } P$ .
- (6) Let  $R$  be a binary relation and  $P$  be a reduction sequence w.r.t.  $R$ . Suppose  $\text{len } P > 1$ . Then there exists a reduction sequence  $Q$  w.r.t.  $R$  such that  $Q \hat{\ } \langle P(\text{len } P) \rangle = P$  and  $\text{len } Q + 1 = \text{len } P$ .
- (7) Let  $R$  be a binary relation and  $P$  be a reduction sequence w.r.t.  $R$ . Suppose  $\text{len } P > 1$ . Then there exists a reduction sequence  $Q$  w.r.t.  $R$  such that  $\text{len } Q + 1 = \text{len } P$  and for every  $k$  such that  $k \in \text{dom } Q$  holds  $Q(k) = P(k + 1)$ .
- (8) For every binary relation  $R$  such that  $\langle x, y \rangle$  is a reduction sequence w.r.t.  $R$  holds  $\langle x, y \rangle \in R$ .
- (9) If  $w = u \hat{\ } v$ , then  $\text{len } u \leq \text{len } w$  and  $\text{len } v \leq \text{len } w$ .
- (10) If  $w = u \hat{\ } v$  and  $u \neq \langle \rangle_E$  and  $v \neq \langle \rangle_E$ , then  $\text{len } u < \text{len } w$  and  $\text{len } v < \text{len } w$ .
- (11) If  $w_1 \hat{\ } v_1 = w_2 \hat{\ } v_2$  and if  $\text{len } w_1 = \text{len } w_2$  or  $\text{len } v_1 = \text{len } v_2$ , then  $w_1 = w_2$  and  $v_1 = v_2$ .
- (12) If  $w_1 \hat{\ } v_1 = w_2 \hat{\ } v_2$  and if  $\text{len } w_1 \leq \text{len } w_2$  or  $\text{len } v_1 \geq \text{len } v_2$ , then there exists  $u$  such that  $w_1 \hat{\ } u = w_2$  and  $v_1 = u \hat{\ } v_2$ .
- (13) If  $w_1 \hat{\ } v_1 = w_2 \hat{\ } v_2$ , then there exists  $u$  such that  $w_1 \hat{\ } u = w_2$  and  $v_1 = u \hat{\ } v_2$  or there exists  $u$  such that  $w_2 \hat{\ } u = w_1$  and  $v_2 = u \hat{\ } v_1$ .

Let us consider  $X$ . We introduce transition-systems over  $X$  which are extensions of 1-sorted structure and are systems

$\langle \text{a carrier, a transition} \rangle$ ,

where the carrier is a set and the transition is a relation between the carrier  $\times X$  and the carrier.

## 2. TRANSITION SYSTEMS OVER SUBSETS OF $E^\omega$

Let us consider  $E, F$  and let  $\mathfrak{T}$  be a transition-system over  $F$ . We say that  $\mathfrak{T}$  is deterministic if and only if the conditions (Def. 1) are satisfied.

- (Def. 1)(i) The transition of  $\mathfrak{T}$  is a function,
- (ii)  $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$ , and
  - (iii) for every element  $s$  of  $\mathfrak{T}$  and for all  $u, v$  such that  $u \neq v$  and  $\langle s, u \rangle \in \text{dom (the transition of } \mathfrak{T})$  and  $\langle s, v \rangle \in \text{dom (the transition of } \mathfrak{T})$  it is not true that there exists  $w$  such that  $u \hat{\ } w = v$  or  $v \hat{\ } w = u$ .

We now state the proposition

- (14) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $\text{dom (the transition of } \mathfrak{T}) = \emptyset$  holds  $\mathfrak{T}$  is deterministic.

Let us consider  $E, F$ . Observe that there exists a transition-system over  $F$  which is strict, non empty, finite, and deterministic.

### 3. PRODUCTIONS

Let us consider  $X$ , let  $\mathfrak{T}$  be a transition-system over  $X$ , and let us consider  $x, y, z$ . The predicate  $x, y \rightarrow_{\mathfrak{T}} z$  is defined by:

(Def. 2)  $\langle \langle x, y \rangle, z \rangle \in$  the transition of  $\mathfrak{T}$ .

We now state several propositions:

- (15) Let  $\mathfrak{T}$  be a transition-system over  $X$ . Suppose  $x, y \rightarrow_{\mathfrak{T}} z$ . Then
  - (i)  $x \in \mathfrak{T}$ ,
  - (ii)  $y \in X$ ,
  - (iii)  $z \in \mathfrak{T}$ ,
  - (iv)  $x \in \text{dom dom}$  (the transition of  $\mathfrak{T}$ ),
  - (v)  $y \in \text{rng dom}$  (the transition of  $\mathfrak{T}$ ), and
  - (vi)  $z \in \text{rng}$  (the transition of  $\mathfrak{T}$ ).
- (16) Let  $\mathfrak{T}_1$  be a transition-system over  $X_1$  and  $\mathfrak{T}_2$  be a transition-system over  $X_2$ . Suppose the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . If  $x, y \rightarrow_{\mathfrak{T}_1} z$ , then  $x, y \rightarrow_{\mathfrak{T}_2} z$ .
- (17) Let  $\mathfrak{T}$  be a transition-system over  $F$ . Suppose the transition of  $\mathfrak{T}$  is a function. If  $x, y \rightarrow_{\mathfrak{T}} z_1$  and  $x, y \rightarrow_{\mathfrak{T}} z_2$ , then  $z_1 = z_2$ .
- (18) For every deterministic transition-system  $\mathfrak{T}$  over  $F$  such that  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ) holds  $x, \langle \rangle_E \not\rightarrow_{\mathfrak{T}} y$ .
- (19) Let  $\mathfrak{T}$  be a deterministic transition-system over  $F$ . If  $u \neq v$  and  $x, u \rightarrow_{\mathfrak{T}} z_1$  and  $x, v \rightarrow_{\mathfrak{T}} z_2$ , then it is not true that there exists  $w$  such that  $u \wedge w = v$  or  $v \wedge w = u$ .

### 4. DIRECT TRANSITIONS

Let us consider  $E, F$ , let  $\mathfrak{T}$  be a transition-system over  $F$ , and let us consider  $x_1, x_2, y_1, y_2$ . The predicate  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$  is defined as follows:

(Def. 3) There exist  $v, w$  such that  $v = y_2$  and  $x_1, w \rightarrow_{\mathfrak{T}} y_1$  and  $x_2 = w \wedge v$ .

The following propositions are true:

- (20) Let  $\mathfrak{T}$  be a transition-system over  $F$ . Suppose  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ . Then  $x_1, y_1 \in \mathfrak{T}$  and  $x_2, y_2 \in E^\omega$  and  $x_1 \in \text{dom dom}$  (the transition of  $\mathfrak{T}$ ) and  $y_1 \in \text{rng}$  (the transition of  $\mathfrak{T}$ ).
- (21) Let  $\mathfrak{T}_1$  be a transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a transition-system over  $F_2$ . Suppose the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$  and  $x_1, x_2 \Rightarrow_{\mathfrak{T}_1} y_1, y_2$ . Then  $x_1, x_2 \Rightarrow_{\mathfrak{T}_2} y_1, y_2$ .

- (22) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $x, u \Rightarrow_{\mathfrak{T}} y, v$  there exists  $w$  such that  $x, w \rightarrow_{\mathfrak{T}} y$  and  $u = w \wedge v$ .
- (23) For every transition-system  $\mathfrak{T}$  over  $F$  holds  $x, y \rightarrow_{\mathfrak{T}} z$  iff  $x, y \Rightarrow_{\mathfrak{T}} z, \langle \rangle_E$ .
- (24) For every transition-system  $\mathfrak{T}$  over  $F$  holds  $x, v \rightarrow_{\mathfrak{T}} y$  iff  $x, v \wedge w \Rightarrow_{\mathfrak{T}} y, w$ .
- (25) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $x, u \Rightarrow_{\mathfrak{T}} y, v$  holds  $x, u \wedge w \Rightarrow_{\mathfrak{T}} y, v \wedge w$ .
- (26) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $x, u \Rightarrow_{\mathfrak{T}} y, v$  holds  $\text{len } u \geq \text{len } v$ .
- (27) Let  $\mathfrak{T}$  be a transition-system over  $F$ . Suppose the transition of  $\mathfrak{T}$  is a function. If  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z$  and  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z$ , then  $y_1 = y_2$ .
- (28) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ) holds  $x, z \not\Rightarrow_{\mathfrak{T}} y, z$ .
- (29) For every transition-system  $\mathfrak{T}$  over  $F$  such that  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ) holds if  $x, u \Rightarrow_{\mathfrak{T}} y, v$ , then  $\text{len } u > \text{len } v$ .
- (30) For every deterministic transition-system  $\mathfrak{T}$  over  $F$  such that  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z_1$  and  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z_2$  holds  $y_1 = y_2$  and  $z_1 = z_2$ .

## 5. REDUCTION RELATION

In the sequel  $\mathfrak{T}$  is a non empty transition-system over  $F$ ,  $s, t$  are elements of  $\mathfrak{T}$ , and  $S$  is a subset of  $\mathfrak{T}$ .

Let us consider  $E, F, \mathfrak{T}$ . The functor  $\Rightarrow_{\mathfrak{T}}$  yielding a binary relation on (the carrier of  $\mathfrak{T}$ )  $\times E^\omega$  is defined as follows:

(Def. 4)  $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$  iff  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ .

The following propositions are true:

- (31) If  $\langle x, y \rangle \in \Rightarrow_{\mathfrak{T}}$ , then there exist  $s, v, t, w$  such that  $x = \langle s, v \rangle$  and  $y = \langle t, w \rangle$ .
- (32) Suppose  $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ . Then  $x_1, y_1 \in \mathfrak{T}$  and  $x_2, y_2 \in E^\omega$  and  $x_1 \in \text{dom dom}$  (the transition of  $\mathfrak{T}$ ) and  $y_1 \in \text{rng}$  (the transition of  $\mathfrak{T}$ ).
- (33) If  $x \in \Rightarrow_{\mathfrak{T}}$ , then there exist  $s, t, v, w$  such that  $x = \langle \langle s, v \rangle, \langle t, w \rangle \rangle$ .
- (34) Let  $\mathfrak{T}_1$  be a non empty transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a non empty transition-system over  $F_2$ . Suppose the carrier of  $\mathfrak{T}_1 =$  the carrier of  $\mathfrak{T}_2$  and the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . Then  $\Rightarrow_{\mathfrak{T}_1} = \Rightarrow_{\mathfrak{T}_2}$ .
- (35) If  $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ , then there exist  $v, w$  such that  $v = y_2$  and  $x_1, w \rightarrow_{\mathfrak{T}} y_1$  and  $x_2 = w \wedge v$ .
- (36) If  $\langle \langle x, u \rangle, \langle y, v \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ , then there exists  $w$  such that  $x, w \rightarrow_{\mathfrak{T}} y$  and  $u = w \wedge v$ .
- (37)  $x, y \rightarrow_{\mathfrak{T}} z$  iff  $\langle \langle x, y \rangle, \langle z, \langle \rangle_E \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ .
- (38)  $x, v \rightarrow_{\mathfrak{T}} y$  iff  $\langle \langle x, v \wedge w \rangle, \langle y, w \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ .

- (39) If  $\langle\langle x, u \rangle, \langle y, v \rangle\rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $\langle\langle x, u \wedge w \rangle, \langle y, v \wedge w \rangle\rangle \in \Rightarrow_{\mathfrak{T}}$ .
- (40) If  $\langle\langle x, u \rangle, \langle y, v \rangle\rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $\text{len } u \geq \text{len } v$ .
- (41) If the transition of  $\mathfrak{T}$  is a function, then if  $\langle x, \langle y_1, z \rangle \rangle, \langle x, \langle y_2, z \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $y_1 = y_2$ .
- (42) If  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ), then if  $\langle\langle x, u \rangle, \langle y, v \rangle\rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $\text{len } u > \text{len } v$ .
- (43) If  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ), then  $\langle\langle x, z \rangle, \langle y, z \rangle\rangle \notin \Rightarrow_{\mathfrak{T}}$ .
- (44) If  $\mathfrak{T}$  is deterministic, then if  $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $y_1 = y_2$ .
- (45) If  $\mathfrak{T}$  is deterministic, then if  $\langle x, \langle y_1, z_1 \rangle \rangle, \langle x, \langle y_2, z_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ , then  $y_1 = y_2$  and  $z_1 = z_2$ .
- (46) If  $\mathfrak{T}$  is deterministic, then  $\Rightarrow_{\mathfrak{T}}$  is function-like.

## 6. REDUCTION SEQUENCES

Let us consider  $x, E$ . The functor  $\text{dim}_2(x, E)$  yields an element of  $E^\omega$  and is defined as follows:

$$\text{(Def. 5)} \quad \text{dim}_2(x, E) = \begin{cases} x_2, & \text{if there exist } y, u \text{ such that } x = \langle y, u \rangle, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state a number of propositions:

- (47) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . If  $k, k+1 \in \text{dom } P$ , then there exist  $s, v, t, w$  such that  $P(k) = \langle s, v \rangle$  and  $P(k+1) = \langle t, w \rangle$ .
- (48) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . If  $k, k+1 \in \text{dom } P$ , then  $P(k) = \langle P(k)_1, P(k)_2 \rangle$  and  $P(k+1) = \langle P(k+1)_1, P(k+1)_2 \rangle$ .
- (49) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . Suppose  $k, k+1 \in \text{dom } P$ . Then
- (i)  $P(k)_1 \in \mathfrak{T}$ ,
  - (ii)  $P(k)_2 \in E^\omega$ ,
  - (iii)  $P(k+1)_1 \in \mathfrak{T}$ ,
  - (iv)  $P(k+1)_2 \in E^\omega$ ,
  - (v)  $P(k)_1 \in \text{dom dom}$  (the transition of  $\mathfrak{T}$ ), and
  - (vi)  $P(k+1)_1 \in \text{rng}$  (the transition of  $\mathfrak{T}$ ).
- (50) Let  $\mathfrak{T}_1$  be a non empty transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a non empty transition-system over  $F_2$ . Suppose the carrier of  $\mathfrak{T}_1 =$  the carrier of  $\mathfrak{T}_2$  and the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . Then every reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}_1}$  is a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}_2}$ .
- (51) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If there exist  $x, u$  such that  $P(1) = \langle x, u \rangle$ , then for every  $k$  such that  $k \in \text{dom } P$  holds  $\text{dim}_2(P(k), E) = P(k)_2$ .
- (52) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(\text{len } P) = \langle y, w \rangle$ , then for every  $k$  such that  $k \in \text{dom } P$  there exists  $u$  such that  $P(k)_2 = u \wedge w$ .

- (53) For every reduction sequence  $P$  w.r.t.  $\Rightarrow_{\mathfrak{T}}$  such that  $P(1) = \langle x, v \rangle$  and  $P(\text{len } P) = \langle y, w \rangle$  there exists  $u$  such that  $v = u \wedge w$ .
- (54) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = \langle x, u \rangle$  and  $P(\text{len } P) = \langle y, u \rangle$ , then for every  $k$  such that  $k \in \text{dom } P$  holds  $P(k)_2 = u$ .
- (55) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . Suppose  $k, k+1 \in \text{dom } P$ . Then there exist  $v, w$  such that  $v = P(k+1)_2$  and  $P(k)_1, w \rightarrow_{\mathfrak{T}} P(k+1)_1$  and  $P(k)_2 = w \wedge v$ .
- (56) Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . Suppose  $k, k+1 \in \text{dom } P$  and  $P(k) = \langle x, u \rangle$  and  $P(k+1) = \langle y, v \rangle$ . Then there exists  $w$  such that  $x, w \rightarrow_{\mathfrak{T}} y$  and  $u = w \wedge v$ .
- (57)  $x, y \rightarrow_{\mathfrak{T}} z$  iff  $\langle \langle x, y \rangle, \langle z, \langle \rangle_E \rangle \rangle$  is a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ .
- (58)  $x, v \rightarrow_{\mathfrak{T}} y$  iff  $\langle \langle x, v \wedge w \rangle, \langle y, w \rangle \rangle$  is a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ .
- (59) For every reduction sequence  $P$  w.r.t.  $\Rightarrow_{\mathfrak{T}}$  such that  $P(1) = \langle x, v \rangle$  and  $P(\text{len } P) = \langle y, w \rangle$  holds  $\text{len } v \geq \text{len } w$ .
- (60) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = \langle x, u \rangle$  and  $P(\text{len } P) = \langle y, u \rangle$ , then  $\text{len } P = 1$  and  $x = y$ .
- (61) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1)_2 = P(\text{len } P)_2$ , then  $\text{len } P = 1$ .
- (62) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = \langle x, u \rangle$  and  $P(\text{len } P) = \langle y, \langle \rangle_E \rangle$ , then  $\text{len } P \leq \text{len } u + 1$ .
- (63) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = \langle x, \langle e \rangle \rangle$  and  $P(\text{len } P) = \langle y, \langle \rangle_E \rangle$ , then  $\text{len } P = 2$ .
- (64) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = \langle x, v \rangle$  and  $P(\text{len } P) = \langle y, w \rangle$ , then  $\text{len } v > \text{len } w$  or  $\text{len } P = 1$  and  $x = y$  and  $v = w$ .
- (65) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k$ . If  $k, k+1 \in \text{dom } P$ , then  $P(k)_2 \neq P(k+1)_2$ .
- (66) Suppose  $\langle \rangle_E \notin \text{rng dom}$  (the transition of  $\mathfrak{T}$ ). Let  $P$  be a reduction sequence w.r.t.  $\Rightarrow_{\mathfrak{T}}$  and given  $k, l$ . If  $k, l \in \text{dom } P$  and  $k < l$ , then  $P(k)_2 \neq P(l)_2$ .
- (67) Suppose  $\mathfrak{T}$  is deterministic. Let  $P, Q$  be reduction sequences w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = Q(1)$ , then for every  $k$  such that  $k \in \text{dom } P$  and  $k \in \text{dom } Q$  holds  $P(k) = Q(k)$ .
- (68) If  $\mathfrak{T}$  is deterministic, then for all reduction sequences  $P, Q$  w.r.t.  $\Rightarrow_{\mathfrak{T}}$  such that  $P(1) = Q(1)$  and  $\text{len } P = \text{len } Q$  holds  $P = Q$ .
- (69) Suppose  $\mathfrak{T}$  is deterministic. Let  $P, Q$  be reduction sequences w.r.t.  $\Rightarrow_{\mathfrak{T}}$ . If  $P(1) = Q(1)$  and  $P(\text{len } P)_2 = Q(\text{len } Q)_2$ , then  $P = Q$ .

## 7. REDUCTIONS

The following propositions are true:

- (70) If  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, v \rangle$  to  $\langle y, w \rangle$ , then there exists  $u$  such that  $v = u \wedge w$ .
- (71) If  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, u \rangle$  to  $\langle y, v \rangle$ , then  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, u \wedge w \rangle$  to  $\langle y, v \wedge w \rangle$ .
- (72) If  $x, y \rightarrow_{\mathfrak{T}} z$ , then  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, y \rangle$  to  $\langle z, \langle \rangle_E \rangle$ .
- (73) If  $x, v \rightarrow_{\mathfrak{T}} y$ , then  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, v \wedge w \rangle$  to  $\langle y, w \rangle$ .
- (74) If  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ , then  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x_1, x_2 \rangle$  to  $\langle y_1, y_2 \rangle$ .
- (75) If  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, v \rangle$  to  $\langle y, w \rangle$ , then  $\text{len } v \geq \text{len } w$ .
- (76) If  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, w \rangle$  to  $\langle y, v \wedge w \rangle$ , then  $v = \langle \rangle_E$ .
- (77) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then if  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, v \rangle$  to  $\langle y, w \rangle$ , then  $\text{len } v > \text{len } w$  or  $x = y$  and  $v = w$ .
- (78) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then if  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, u \rangle$  to  $\langle y, u \rangle$ , then  $x = y$ .
- (79) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then if  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x, \langle e \rangle \rangle$  to  $\langle y, \langle \rangle_E \rangle$ , then  $\langle \langle x, \langle e \rangle \rangle, \langle y, \langle \rangle_E \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ .
- (80) If  $\mathfrak{T}$  is deterministic, then if  $\Rightarrow_{\mathfrak{T}}$  reduces  $x$  to  $\langle y_1, z \rangle$  and  $\Rightarrow_{\mathfrak{T}}$  reduces  $x$  to  $\langle y_2, z \rangle$ , then  $y_1 = y_2$ .

## 8. TRANSITIONS

Let us consider  $E, F, \mathfrak{T}, x_1, x_2, y_1, y_2$ . The predicate  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$  is defined as follows:

(Def. 6)  $\Rightarrow_{\mathfrak{T}}$  reduces  $\langle x_1, x_2 \rangle$  to  $\langle y_1, y_2 \rangle$ .

We now state a number of propositions:

- (81) Let  $\mathfrak{T}_1$  be a non empty transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a non empty transition-system over  $F_2$ . Suppose the carrier of  $\mathfrak{T}_1 =$  the carrier of  $\mathfrak{T}_2$  and the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . If  $x_1, x_2 \Rightarrow_{\mathfrak{T}_1}^* y_1, y_2$ , then  $x_1, x_2 \Rightarrow_{\mathfrak{T}_2}^* y_1, y_2$ .
- (82)  $x, y \Rightarrow_{\mathfrak{T}}^* x, y$ .
- (83) If  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$  and  $y_1, y_2 \Rightarrow_{\mathfrak{T}}^* z_1, z_2$ , then  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* z_1, z_2$ .
- (84) If  $x, y \rightarrow_{\mathfrak{T}} z$ , then  $x, y \Rightarrow_{\mathfrak{T}}^* z, \langle \rangle_E$ .
- (85) If  $x, v \rightarrow_{\mathfrak{T}} y$ , then  $x, v \wedge w \Rightarrow_{\mathfrak{T}}^* y, w$ .
- (86) If  $x, u \Rightarrow_{\mathfrak{T}}^* y, v$ , then  $x, u \wedge w \Rightarrow_{\mathfrak{T}}^* y, v \wedge w$ .
- (87) If  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ , then  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$ .
- (88) If  $x, v \Rightarrow_{\mathfrak{T}}^* y, w$ , then there exists  $u$  such that  $v = u \wedge w$ .
- (89) If  $x, v \Rightarrow_{\mathfrak{T}}^* y, w$ , then  $\text{len } w \leq \text{len } v$ .
- (90) If  $x, w \Rightarrow_{\mathfrak{T}}^* y, v \wedge w$ , then  $v = \langle \rangle_E$ .

- (91) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then  $x, u \Rightarrow_{\mathfrak{T}}^* y, u$  iff  $x = y$ .
- (92) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then if  $x, \langle e \rangle \Rightarrow_{\mathfrak{T}}^* y, \langle \rangle_E$ , then  $x, \langle e \rangle \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$ .
- (93) If  $\mathfrak{T}$  is deterministic, then if  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, z$  and  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_2, z$ , then  $y_1 = y_2$ .

## 9. ACCEPTANCE OF WORDS

Let us consider  $E, F, \mathfrak{T}, x_1, x_2, y$ . The predicate  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y$  is defined as follows:

(Def. 7)  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y, \langle \rangle_E$ .

We now state several propositions:

- (94) Let  $\mathfrak{T}_1$  be a non empty transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a non empty transition-system over  $F_2$ . Suppose the carrier of  $\mathfrak{T}_1 =$  the carrier of  $\mathfrak{T}_2$  and the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . If  $x, y \Rightarrow_{\mathfrak{T}_1}^* z$ , then  $x, y \Rightarrow_{\mathfrak{T}_2}^* z$ .
- (95)  $x, \langle \rangle_E \Rightarrow_{\mathfrak{T}}^* x$ .
- (96) If  $x, u \Rightarrow_{\mathfrak{T}}^* y$ , then  $x, u \wedge v \Rightarrow_{\mathfrak{T}}^* y, v$ .
- (97) If  $x, y \rightarrow_{\mathfrak{T}} z$ , then  $x, y \Rightarrow_{\mathfrak{T}}^* z$ .
- (98) If  $x_1, x_2 \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$ , then  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y$ .
- (99) If  $x, u \Rightarrow_{\mathfrak{T}}^* y$  and  $y, v \Rightarrow_{\mathfrak{T}}^* z$ , then  $x, u \wedge v \Rightarrow_{\mathfrak{T}}^* z$ .
- (100) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then  $x, \langle \rangle_E \Rightarrow_{\mathfrak{T}}^* y$  iff  $x = y$ .
- (101) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then if  $x, \langle e \rangle \Rightarrow_{\mathfrak{T}}^* y$ , then  $x, \langle e \rangle \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$ .
- (102) If  $\mathfrak{T}$  is deterministic, then if  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1$  and  $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_2$ , then  $y_1 = y_2$ .

## 10. REACHABLE STATES

Let us consider  $E, F, \mathfrak{T}, x, X$ . The functor  $x\text{-succ}_{\mathfrak{T}}(X)$  yields a subset of  $\mathfrak{T}$  and is defined as follows:

(Def. 8)  $x\text{-succ}_{\mathfrak{T}}(X) = \{s : \bigvee_t (t \in X \wedge t, x \Rightarrow_{\mathfrak{T}}^* s)\}$ .

The following propositions are true:

- (103)  $s \in x\text{-succ}_{\mathfrak{T}}(X)$  iff there exists  $t$  such that  $t \in X$  and  $t, x \Rightarrow_{\mathfrak{T}}^* s$ .
- (104) If  $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ , then  $\langle \rangle_E\text{-succ}_{\mathfrak{T}}(S) = S$ .
- (105) Let  $\mathfrak{T}_1$  be a non empty transition-system over  $F_1$  and  $\mathfrak{T}_2$  be a non empty transition-system over  $F_2$ . Suppose the carrier of  $\mathfrak{T}_1 =$  the carrier of  $\mathfrak{T}_2$  and the transition of  $\mathfrak{T}_1 =$  the transition of  $\mathfrak{T}_2$ . Then  $x\text{-succ}_{\mathfrak{T}_1}(X) = x\text{-succ}_{\mathfrak{T}_2}(X)$ .



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