

Riemann Integral of Functions from \mathbb{R} into \mathcal{R}^n

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Summary. In this article, we define the Riemann Integral of functions from \mathbb{R} into \mathcal{R}^n , and prove the linearity of this operator. The presented method is based on [21].

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The articles [22], [1], [23], [5], [6], [15], [20], [24], [7], [17], [16], [2], [4], [3], [8], [18], [9], [12], [10], [14], [13], [19], and [11] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathbb{R} , let S be a non empty Division of A , and let D be an element of S . A finite sequence of elements of \mathbb{R} is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\text{len } \text{it} = \text{len } D$, and

(ii) for every natural number i such that $i \in \text{dom } D$ there exists an element r of \mathbb{R} such that $r \in \text{rng}(f \upharpoonright \text{divset}(D, i))$ and $\text{it}(i) = r \cdot \text{vol}(\text{divset}(D, i))$.

Let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathbb{R} , let S be a non empty Division of A , let D be an element of S , and let F be a middle volume of f and D . The functor $\text{middle_sum}(f, F)$ yielding a real number is defined as follows:

(Def. 2) $\text{middle_sum}(f, F) = \sum F$.

We now state four propositions:

- (1) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , S be a non empty Division of A , D be an element of S , and F be a middle volume of f and D . If $f|_A$ is lower bounded, then $\text{lower_sum}(f, D) \leq \text{middle_sum}(f, F)$.
- (2) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , S be a non empty Division of A , D be an element of S , and F be a middle volume of f and D . If $f|_A$ is upper bounded, then $\text{middle_sum}(f, F) \leq \text{upper_sum}(f, D)$.
- (3) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , S be a non empty Division of A , D be an element of S , and e be a real number. Suppose $f|_A$ is lower bounded and $0 < e$. Then there exists a middle volume F of f and D such that $\text{middle_sum}(f, F) \leq \text{lower_sum}(f, D) + e$.
- (4) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , S be a non empty Division of A , D be an element of S , and e be a real number. Suppose $f|_A$ is upper bounded and $0 < e$. Then there exists a middle volume F of f and D such that $\text{upper_sum}(f, D) - e \leq \text{middle_sum}(f, F)$.

Let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathbb{R} , and let T be a DivSequence of A . A function from \mathbb{N} into \mathbb{R}^* is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of \mathbb{N} holds $\text{it}(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathbb{R} , let T be a DivSequence of A , let S be a middle volume sequence of f and T , and let k be an element of \mathbb{N} . Then $S(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathbb{R} , let T be a DivSequence of A , and let S be a middle volume sequence of f and T . The functor $\text{middle_sum}(f, S)$ yields a sequence of real numbers and is defined by:

(Def. 4) For every element i of \mathbb{N} holds $(\text{middle_sum}(f, S))(i) = \text{middle_sum}(f, S(i))$.

We now state several propositions:

- (5) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , T be a DivSequence of A , S be a middle volume sequence of f and T , and i be an element of \mathbb{N} . If $f|_A$ is lower bounded, then $(\text{lower_sum}(f, T))(i) \leq (\text{middle_sum}(f, S))(i)$.
- (6) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , T be a DivSequence of A , S be a middle volume sequence of f and T , and i be an element of \mathbb{N} . If $f|_A$ is upper bounded, then $(\text{middle_sum}(f, S))(i) \leq (\text{upper_sum}(f, T))(i)$.
- (7) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , T be a DivSequence of A , and e be an element of \mathbb{R} . Suppose $0 < e$ and $f|_A$ is lower bounded. Then there exists a middle volume sequence S of

f and T such that for every element i of \mathbb{N} holds $(\text{middle_sum}(f, S))(i) \leq (\text{lower_sum}(f, T))(i) + e$.

- (8) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , T be a DivSequence of A , and e be an element of \mathbb{R} . Suppose $0 < e$ and $f \upharpoonright A$ is upper bounded. Then there exists a middle volume sequence S of f and T such that for every element i of \mathbb{N} holds $(\text{upper_sum}(f, T))(i) - e \leq (\text{middle_sum}(f, S))(i)$.
- (9) Let A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathbb{R} , T be a DivSequence of A , and S be a middle volume sequence of f and T . Suppose f is bounded and f is integrable on A and δ_T is convergent and $\lim(\delta_T) = 0$. Then $\text{middle_sum}(f, S)$ is convergent and $\lim \text{middle_sum}(f, S) = \text{integral } f$.
- (10) Let A be a closed-interval subset of \mathbb{R} and f be a function from A into \mathbb{R} . Suppose f is bounded. Then f is integrable on A if and only if there exists a real number I such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\text{middle_sum}(f, S)$ is convergent and $\lim \text{middle_sum}(f, S) = I$.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathcal{R}^n , let S be a non empty Division of A , and let D be an element of S . A finite sequence of elements of \mathcal{R}^n is said to be a middle volume of f and D if it satisfies the conditions (Def. 5).

- (Def. 5)(i) $\text{len } it = \text{len } D$, and
- (ii) for every natural number i such that $i \in \text{dom } D$ there exists an element r of \mathcal{R}^n such that $r \in \text{rng}(f \upharpoonright \text{divset}(D, i))$ and $it(i) = \text{vol}(\text{divset}(D, i)) \cdot r$.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathcal{R}^n , let S be a non empty Division of A , let D be an element of S , and let F be a middle volume of f and D . The functor $\text{middle_sum}(f, F)$ yielding an element of \mathcal{R}^n is defined by the condition (Def. 6).

- (Def. 6) Let i be an element of \mathbb{N} . Suppose $i \in \text{Seg } n$. Then there exists a finite sequence F_1 of elements of \mathbb{R} such that $F_1 = \text{proj}(i, n) \cdot F$ and $(\text{middle_sum}(f, F))(i) = \sum F_1$.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathcal{R}^n , and let T be a DivSequence of A . A function from \mathbb{N} into $(\mathcal{R}^n)^*$ is said to be a middle volume sequence of f and T if:

- (Def. 7) For every element k of \mathbb{N} holds $it(k)$ is a middle volume of f and $T(k)$.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , let f be a function from A into \mathcal{R}^n , let T be a DivSequence of A , let S be a middle volume sequence of f and T , and let k be an element of \mathbb{N} . Then $S(k)$ is a middle volume of f and $T(k)$.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , let f be a

function from A into \mathcal{R}^n , let T be a DivSequence of A , and let S be a middle volume sequence of f and T . The functor $\text{middle_sum}(f, S)$ yields a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and is defined as follows:

(Def. 8) For every element i of \mathbb{N} holds $(\text{middle_sum}(f, S))(i) = \text{middle_sum}(f, S(i))$.

Let n be an element of \mathbb{N} , let Z be a non empty set, and let f, g be partial functions from Z to \mathcal{R}^n . The functor $f + g$ yielding a partial function from Z to \mathcal{R}^n is defined by:

(Def. 9) $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$ and for every element c of Z such that $c \in \text{dom}(f + g)$ holds $(f + g)_c = f_c + g_c$.

The functor $f - g$ yielding a partial function from Z to \mathcal{R}^n is defined as follows:

(Def. 10) $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$ and for every element c of Z such that $c \in \text{dom}(f - g)$ holds $(f - g)_c = f_c - g_c$.

Let n be an element of \mathbb{N} , let r be a real number, let Z be a non empty set, and let f be a partial function from Z to \mathcal{R}^n . The functor $r f$ yielding a partial function from Z to \mathcal{R}^n is defined as follows:

(Def. 11) $\text{dom}(r f) = \text{dom } f$ and for every element c of Z such that $c \in \text{dom}(r f)$ holds $(r f)_c = r \cdot f_c$.

2. DEFINITION OF RIEMANN INTEGRAL OF FUNCTIONS FROM \mathbb{R} INTO \mathcal{R}^n

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , and let f be a function from A into \mathcal{R}^n . We say that f is bounded if and only if:

(Def. 12) For every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $\text{proj}(i, n) \cdot f$ is bounded.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , and let f be a function from A into \mathcal{R}^n . We say that f is integrable if and only if:

(Def. 13) For every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $\text{proj}(i, n) \cdot f$ is integrable on A .

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , and let f be a function from A into \mathcal{R}^n . The functor $\text{integral } f$ yielding an element of \mathcal{R}^n is defined by:

(Def. 14) $\text{dom integral } f = \text{Seg } n$ and for every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $(\text{integral } f)(i) = \text{integral proj}(i, n) \cdot f$.

One can prove the following two propositions:

(11) Let n be an element of \mathbb{N} , A be a closed-interval subset of \mathbb{R} , f be a function from A into \mathcal{R}^n , T be a DivSequence of A , and S be a middle volume sequence of f and T . Suppose f is bounded and integrable and δ_T is convergent and $\lim(\delta_T) = 0$. Then $\text{middle_sum}(f, S)$ is convergent and $\lim \text{middle_sum}(f, S) = \text{integral } f$.

- (12) Let n be an element of \mathbb{N} , A be a closed-interval subset of \mathbb{R} , and f be a function from A into \mathcal{R}^n . Suppose f is bounded. Then f is integrable if and only if there exists an element I of \mathcal{R}^n such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\text{middle_sum}(f, S)$ is convergent and $\lim \text{middle_sum}(f, S) = I$.

Let n be an element of \mathbb{N} and let f be a partial function from \mathbb{R} to \mathcal{R}^n . We say that f is bounded if and only if:

- (Def. 15) For every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $\text{proj}(i, n) \cdot f$ is bounded.

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , and let f be a partial function from \mathbb{R} to \mathcal{R}^n . We say that f is integrable on A if and only if:

- (Def. 16) For every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $\text{proj}(i, n) \cdot f$ is integrable on A .

Let n be an element of \mathbb{N} , let A be a closed-interval subset of \mathbb{R} , and let f be a partial function from \mathbb{R} to \mathcal{R}^n . The functor $\int_A f(x)dx$ yields an element of \mathcal{R}^n and is defined by:

- (Def. 17) $\text{dom} \int_A f(x)dx = \text{Seg } n$ and for every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $(\int_A f(x)dx)(i) = \int_A (\text{proj}(i, n) \cdot f)(x)dx$.

The following two propositions are true:

- (13) Let n be an element of \mathbb{N} , A be a closed-interval subset of \mathbb{R} , f be a partial function from \mathbb{R} to \mathcal{R}^n , and g be a function from A into \mathcal{R}^n . Suppose $f \upharpoonright A = g$. Then f is integrable on A if and only if g is integrable.
- (14) Let n be an element of \mathbb{N} , A be a closed-interval subset of \mathbb{R} , f be a partial function from \mathbb{R} to \mathcal{R}^n , and g be a function from A into \mathcal{R}^n . If $f \upharpoonright A = g$, then $\int_A f(x)dx = \text{integral } g$.

Let a, b be real numbers, let n be an element of \mathbb{N} , and let f be a partial function from \mathbb{R} to \mathcal{R}^n . The functor $\int_a^b f(x)dx$ yielding an element of \mathcal{R}^n is defined as follows:

- (Def. 18) $\text{dom} \int_a^b f(x)dx = \text{Seg } n$ and for every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $(\int_a^b f(x)dx)(i) = \int_a^b (\text{proj}(i, n) \cdot f)(x)dx$.

3. LINEARITY OF INTEGRATION OPERATOR

We now state several propositions:

- (15) Let n be an element of \mathbb{N} , f_1, f_2 be partial functions from \mathbb{R} to \mathcal{R}^n , and i be an element of \mathbb{N} . If $i \in \text{Seg } n$, then $\text{proj}(i, n) \cdot (f_1 + f_2) = \text{proj}(i, n) \cdot f_1 + \text{proj}(i, n) \cdot f_2$ and $\text{proj}(i, n) \cdot (f_1 - f_2) = \text{proj}(i, n) \cdot f_1 - \text{proj}(i, n) \cdot f_2$.
- (16) Let n be an element of \mathbb{N} , r be a real number, f be a partial function from \mathbb{R} to \mathcal{R}^n , and i be an element of \mathbb{N} . If $i \in \text{Seg } n$, then $\text{proj}(i, n) \cdot (r f) = r (\text{proj}(i, n) \cdot f)$.
- (17) Let n be an element of \mathbb{N} , A be a closed-interval subset of \mathbb{R} , and f_1, f_2 be partial functions from \mathbb{R} to \mathcal{R}^n . Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \text{dom } f_1$ and $A \subseteq \text{dom } f_2$ and $f_1 \upharpoonright A$ is bounded and $f_2 \upharpoonright A$ is bounded. Then $f_1 + f_2$ is integrable on A and $f_1 - f_2$ is integrable on A and $\int_A (f_1 + f_2)(x) dx = \int_A f_1(x) dx + \int_A f_2(x) dx$ and $\int_A (f_1 - f_2)(x) dx = \int_A f_1(x) dx - \int_A f_2(x) dx$.
- (18) Let n be an element of \mathbb{N} , r be a real number, A be a closed-interval subset of \mathbb{R} , and f be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose $A \subseteq \text{dom } f$ and f is integrable on A and $f \upharpoonright A$ is bounded. Then $r f$ is integrable on A and $\int_A (r f)(x) dx = r \cdot \int_A f(x) dx$.
- (19) Let n be an element of \mathbb{N} , f be a partial function from \mathbb{R} to \mathcal{R}^n , A be a closed-interval subset of \mathbb{R} , and a, b be real numbers. If $A = [a, b]$, then $\int_A f(x) dx = \int_a^b f(x) dx$.
- (20) Let n be an element of \mathbb{N} , f be a partial function from \mathbb{R} to \mathcal{R}^n , A be a closed-interval subset of \mathbb{R} , and a, b be real numbers. If $A = [b, a]$, then $-\int_A f(x) dx = \int_a^b f(x) dx$.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(4):643–649, 1990.
- [4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.

- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(4):661–668, 1990.
- [9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [10] Noboru Endou and Artur Kornilowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(1):93–102, 1999.
- [11] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(4):577–580, 2005.
- [12] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces \mathcal{R}^n . *Formalized Mathematics*, 15(2):65–72, 2007, doi:10.2478/v10037-007-0008-5.
- [13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from \mathbb{R} to \mathbb{R} and integrability for continuous functions. *Formalized Mathematics*, 9(2):281–284, 2001.
- [14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Formalized Mathematics*, 9(1):191–196, 2001.
- [15] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [16] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [17] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [19] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [20] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [21] Murray R. Spiegel. *Theory and Problems of Vector Analysis*. McGraw-Hill, 1974.
- [22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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