

Basic Properties of Even and Odd Functions

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Summary. In this article we present definitions, basic properties and some examples of even and odd functions [6].

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The articles [2], [5], [1], [8], [14], [12], [15], [7], [17], [3], [4], [11], [19], [13], [10], [18], [16], and [9] provide the notation and terminology for this paper.

1. EVEN AND ODD FUNCTIONS

In this paper x, r denote real numbers.

Let A be a set. We say that A is symmetrical if and only if:

(Def. 1) For every complex number x such that $x \in A$ holds $-x \in A$.

One can check that there exists a subset of \mathbb{C} which is symmetrical.

Let us note that there exists a subset of \mathbb{R} which is symmetrical.

In the sequel A is a symmetrical subset of \mathbb{C} .

Let R be a binary relation. We say that R has symmetrical domain if and only if:

(Def. 2) $\text{dom } R$ is symmetrical.

Let us observe that every binary relation which is empty has also symmetrical domain and there exists a binary relation which has symmetrical domain.

Let R be a binary relation with symmetrical domain. One can check that $\text{dom } R$ is symmetrical.

Let X, Y be complex-membered sets and let F be a partial function from X to Y . We say that F is quasi even if and only if:

(Def. 3) For every x such that $x, -x \in \text{dom } F$ holds $F(-x) = F(x)$.

Let X, Y be complex-membered sets and let F be a partial function from X to Y . We say that F is even if and only if:

(Def. 4) F is quasi even and has symmetrical domain.

Let X, Y be complex-membered sets. Note that every partial function from X to Y which is quasi even and has symmetrical domain is also even and every partial function from X to Y which is even is also quasi even and has symmetrical domain.

Let A be a set, let X, Y be complex-membered sets, and let F be a partial function from X to Y . We say that F is even on A if and only if:

(Def. 5) $A \subseteq \text{dom } F$ and $F \upharpoonright A$ is even.

Let X, Y be complex-membered sets and let F be a partial function from X to Y . We say that F is quasi odd if and only if:

(Def. 6) For every x such that $x, -x \in \text{dom } F$ holds $F(-x) = -F(x)$.

Let X, Y be complex-membered sets and let F be a partial function from X to Y . We say that F is odd if and only if:

(Def. 7) F is quasi odd and has symmetrical domain.

Let X, Y be complex-membered sets. Note that every partial function from X to Y which is quasi odd and has symmetrical domain is also odd and every partial function from X to Y which is odd is also quasi odd and has symmetrical domain.

Let A be a set, let X, Y be complex-membered sets, and let F be a partial function from X to Y . We say that F is odd on A if and only if:

(Def. 8) $A \subseteq \text{dom } F$ and $F \upharpoonright A$ is odd.

In the sequel F, G denote partial functions from \mathbb{R} to \mathbb{R} .

One can prove the following propositions:

- (1) F is odd on A iff $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $F(x) + F(-x) = 0$.
- (2) F is even on A iff $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $F(x) - F(-x) = 0$.
- (3) If F is odd on A and for every x such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$.
- (4) If $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$, then F is odd on A .
- (5) If F is even on A and for every x such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$.
- (6) If $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$, then F is even on A .

- (7) If F is even on A and odd on A , then for every x such that $x \in A$ holds $F(x) = 0$.
- (8) If F is even on A , then for every x such that $x \in A$ holds $F(x) = F(|x|)$.
- (9) If $A \subseteq \text{dom } F$ and for every x such that $x \in A$ holds $F(x) = F(|x|)$, then F is even on A .
- (10) If F is odd on A and G is odd on A , then $F + G$ is odd on A .
- (11) If F is even on A and G is even on A , then $F + G$ is even on A .
- (12) If F is odd on A and G is odd on A , then $F - G$ is odd on A .
- (13) If F is even on A and G is even on A , then $F - G$ is even on A .
- (14) If F is odd on A , then rF is odd on A .
- (15) If F is even on A , then rF is even on A .
- (16) If F is odd on A , then $-F$ is odd on A .
- (17) If F is even on A , then $-F$ is even on A .
- (18) If F is odd on A , then F^{-1} is odd on A .
- (19) If F is even on A , then F^{-1} is even on A .
- (20) If F is odd on A , then $|F|$ is even on A .
- (21) If F is even on A , then $|F|$ is even on A .
- (22) If F is odd on A and G is odd on A , then FG is even on A .
- (23) If F is even on A and G is even on A , then FG is even on A .
- (24) If F is even on A and G is odd on A , then FG is odd on A .
- (25) If F is even on A , then $r + F$ is even on A .
- (26) If F is even on A , then $F - r$ is even on A .
- (27) If F is even on A , then F^2 is even on A .
- (28) If F is odd on A , then F^2 is even on A .
- (29) If F is odd on A and G is odd on A , then F/G is even on A .
- (30) If F is even on A and G is even on A , then F/G is even on A .
- (31) If F is odd on A and G is even on A , then F/G is odd on A .
- (32) If F is even on A and G is odd on A , then F/G is odd on A .
- (33) If F is odd, then $-F$ is odd.
- (34) If F is even, then $-F$ is even.
- (35) If F is odd, then F^{-1} is odd.
- (36) If F is even, then F^{-1} is even.
- (37) If F is odd, then $|F|$ is even.
- (38) If F is even, then $|F|$ is even.
- (39) If F is odd, then F^2 is even.
- (40) If F is even, then F^2 is even.
- (41) If F is even, then $r + F$ is even.

- (42) If F is even, then $F - r$ is even.
- (43) If F is odd, then $r F$ is odd.
- (44) If F is even, then $r F$ is even.
- (45) If F is odd and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F + G$ is odd.
- (46) If F is even and G is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F + G$ is even.
- (47) If F is odd and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F - G$ is odd.
- (48) If F is even and G is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F - G$ is even.
- (49) If F is odd and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F G$ is even.
- (50) If F is even and G is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F G$ is even.
- (51) If F is even and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then $F G$ is odd.
- (52) If F is odd and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then F/G is even.
- (53) If F is even and G is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then F/G is even.
- (54) If F is odd and G is even and $\text{dom } F \cap \text{dom } G$ is symmetrical, then F/G is odd.
- (55) If F is even and G is odd and $\text{dom } F \cap \text{dom } G$ is symmetrical, then F/G is odd.

2. SOME EXAMPLES

The function signum from \mathbb{R} into \mathbb{R} is defined by:

(Def. 9) For every real number x holds $\text{signum}(x) = \text{sgn } x$.

Let x be a real number. One can verify that $\text{signum}(x)$ is real.

Next we state a number of propositions:

- (56) For every real number x such that $x > 0$ holds $\text{signum}(x) = 1$.
- (57) For every real number x such that $x < 0$ holds $\text{signum}(x) = -1$.
- (58) $\text{signum}(0) = 0$.
- (59) For every real number x holds $\text{signum}(-x) = -\text{signum}(x)$.
- (60) For every symmetrical subset A of \mathbb{R} holds signum is odd on A .
- (61) For every real number x such that $x \geq 0$ holds $|\square|_{\mathbb{R}}(x) = x$.

- (62) For every real number x such that $x < 0$ holds $|\square|_{\mathbb{R}}(x) = -x$.
- (63) For every real number x holds $|\square|_{\mathbb{R}}(-x) = |\square|_{\mathbb{R}}(x)$.
- (64) For every symmetrical subset A of \mathbb{R} holds $|\square|_{\mathbb{R}}$ is even on A .
- (65) For every symmetrical subset A of \mathbb{R} holds the function \sin is odd on A .
- (66) For every symmetrical subset A of \mathbb{R} holds the function \cos is even on A .

Let us observe that the function \sin is odd.

Let us observe that the function \cos is even.

We now state two propositions:

- (67) For every symmetrical subset A of \mathbb{R} holds the function \sinh is odd on A .
- (68) For every symmetrical subset A of \mathbb{R} holds the function \cosh is even on A .

Let us note that the function \sinh is odd.

Let us mention that the function \cosh is even.

The following propositions are true:

- (69) If $A \subseteq]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function \tan is odd on A .
- (70) Suppose $A \subseteq \text{dom}(\text{the function } \tan)$ and for every x such that $x \in A$ holds $(\text{the function } \cos)(x) \neq 0$. Then the function \tan is odd on A .
- (71) Suppose $A \subseteq \text{dom}(\text{the function } \cot)$ and for every x such that $x \in A$ holds $(\text{the function } \sin)(x) \neq 0$. Then the function \cot is odd on A .
- (72) If $A \subseteq [-1, 1]$, then the function \arctan is odd on A .
- (73) For every symmetrical subset A of \mathbb{R} holds $|\text{the function } \sin|$ is even on A .
- (74) For every symmetrical subset A of \mathbb{R} holds $|\text{the function } \cos|$ is even on A .
- (75) For every symmetrical subset A of \mathbb{R} holds $(\text{the function } \sin)^{-1}$ is odd on A .
- (76) For every symmetrical subset A of \mathbb{R} holds $(\text{the function } \cos)^{-1}$ is even on A .
- (77) For every symmetrical subset A of \mathbb{R} holds $-\text{the function } \sin$ is odd on A .
- (78) For every symmetrical subset A of \mathbb{R} holds $-\text{the function } \cos$ is even on A .
- (79) For every symmetrical subset A of \mathbb{R} holds $(\text{the function } \sin)^2$ is even on A .
- (80) For every symmetrical subset A of \mathbb{R} holds $(\text{the function } \cos)^2$ is even on A .

In the sequel B denotes a symmetrical subset of \mathbb{R} .

One can prove the following propositions:

- (81) If $B \subseteq \text{dom}(\text{the function sec})$, then the function sec is even on B .
- (82) If for every real number x such that $x \in B$ holds $(\text{the function cos})(x) \neq 0$, then the function sec is even on B .
- (83) If $B \subseteq \text{dom}(\text{the function cosec})$, then the function cosec is odd on B .
- (84) If for every real number x such that $x \in B$ holds $(\text{the function sin})(x) \neq 0$, then the function cosec is odd on B .

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