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Equivalence of Deterministic and Nondeterministic Epsilon Automata

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Summary. Based on concepts introduced in [14], semiautomata and left-languages, automata and right-languages, and langauges accepted by automata are defined. The powerset construction is defined for transition systems, semiautomata and automata. Finally, the equivalence of deterministic and nondeterministic epsilon automata is shown.

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The terminology and notation used in this paper have been introduced in the following articles: [1], [8], [2], [11], [6], [18], [7], [9], [17], [16], [15], [4], [10], [13], [3], [12], [5], and [14].

1. Preliminaries

For simplicity, we adopt the following convention: x, y, X denote sets, E denotes a non empty set, e denotes an element of E, u, u_1, v, v_1, v_2, w denote elements of E^ω, F denotes a subset of E^ω, i, k, l denote natural numbers, $\mathfrak{T}$ denotes a non empty transition-system over F, and S, T denote subsets of $\mathfrak{T}$.

One can prove the following propositions:

(1) If $i \geq k + l$, then $i \geq k$.

(2) For all finite sequences a, b such that $a \mathbin{^\frown} b = a$ or $b \mathbin{^\frown} a = a$ holds $b = \emptyset$.

(3) For all finite sequences p, q such that $k \in \operatorname{dom} p$ and $\operatorname{len} p + 1 = \operatorname{len} q$ holds $k + 1 \in \operatorname{dom} q$.

(4) If $\operatorname{len} u = 1$, then there exists e such that $\langle e \rangle = u$ and $e = u(0)$.

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(5) If $k \neq 0$ and $\operatorname{len} u \leq k+1$, then there exist v_1, v_2 such that $\operatorname{len} v_1 \leq k$ and $\operatorname{len} v_2 \leq k$ and $u = v_1 \frown v_2$.

(6) For all finite 0-sequences p, q such that $\langle x \rangle \frown p = \langle y \rangle \frown q$ holds $x = y$ and $p = q$.

(7) If $\operatorname{len} u > 0$, then there exist e, u_1 such that $u = \langle e \rangle \frown u_1$.

Let us consider E. One can verify that $\operatorname{Lex} E$ is non empty.

Next we state three propositions:

(8) $\langle\rangle_E \notin \operatorname{Lex} E$.

(9) $u \in \operatorname{Lex} E$ iff $\operatorname{len} u = 1$.

(10) If $u \neq v$ and u, $v \in \operatorname{Lex} E$, then it is not true that there exists w such that $u \frown w = v$ or $w \frown u = v$.

2. Transition Systems over Lex E

The following propositions are true:

(11) For every transition-system $\mathfrak{T}$ over $\operatorname{Lex} E$ holds $\langle\rangle_E \notin \operatorname{rng} \operatorname{dom}$ (the transition of $\mathfrak{T}$).

(12) For every transition-system $\mathfrak{T}$ over $\operatorname{Lex} E$ such that the transition of $\mathfrak{T}$ is a function holds $\mathfrak{T}$ is deterministic.

3. Powerset Construction for Transition Systems

Let us consider E, F, $\mathfrak{T}$. The functor $\operatorname{bool} \mathfrak{T}$ yielding a strict transition-system over $\operatorname{Lex} E$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\operatorname{bool} \mathfrak{T} = 2^{\text{the carrier of } \mathfrak{T}}$, and

(ii) for all S, w, T holds $\langle\langle S, w\rangle, T\rangle \in$ the transition of $\operatorname{bool} \mathfrak{T}$ iff $\operatorname{len} w = 1$ and $T = w\text{-succ}_{\mathfrak{T}}(S)$.

Let us consider E, F, $\mathfrak{T}$. Note that $\operatorname{bool} \mathfrak{T}$ is non empty and deterministic.

Let us consider E, F and let $\mathfrak{T}$ be a finite non empty transition-system over F. One can check that $\operatorname{bool} \mathfrak{T}$ is finite.

The following two propositions are true:

(13) If $x, \langle e \rangle \Rightarrow^*_{\operatorname{bool} \mathfrak{T}} y, \langle\rangle_E$, then $x, \langle e \rangle \Rightarrow_{\operatorname{bool} \mathfrak{T}} y, \langle\rangle_E$.

(14) If $\operatorname{len} w = 1$, then $X = w\text{-succ}_{\mathfrak{T}}(S)$ iff $S, w \Rightarrow^*_{\operatorname{bool} \mathfrak{T}} X$.

4. Semiautomata

Let us consider E, F. We consider semiautomata over F as extensions of transition-system over F as systems

⟨ a carrier, a transition, an initial state ⟩,

where the carrier is a set, the transition is a relation between the carrier$\times F$ and the carrier, and the initial state is a subset of the carrier.

Let us consider E, F and let $\mathfrak{S}$ be a semiautomaton over F. We say that $\mathfrak{S}$ is deterministic if and only if:

(Def. 2) The transition-system of $\mathfrak{S}$ is deterministic and Card (the initial state of $\mathfrak{S}$) $= 1$.

Let us consider E, F. One can check that there exists a semiautomaton over F which is strict, non empty, finite, and deterministic.

In the sequel $\mathfrak{S}$ is a non empty semiautomaton over F.

Let us consider E, F, $\mathfrak{S}$. Observe that the transition-system of $\mathfrak{S}$ is non empty.

Let us consider E, F, $\mathfrak{S}$. The functor bool $\mathfrak{S}$ yields a strict semiautomaton over Lex E and is defined by the conditions (Def. 3).

(Def. 3)(i) The transition-system of bool $\mathfrak{S}$ = bool (the transition-system of $\mathfrak{S}$), and

(ii) the initial state of bool $\mathfrak{S} = \{\langle\rangle_E\text{-succ}_{\mathfrak{S}}(\text{the initial state of } \mathfrak{S})\}$.

Let us consider E, F, $\mathfrak{S}$. Observe that bool $\mathfrak{S}$ is non empty and deterministic.

The following proposition is true

(15) The carrier of bool $\mathfrak{S} = 2^{\text{the carrier of } \mathfrak{S}}$.

Let us consider E, F and let $\mathfrak{S}$ be a finite non empty semiautomaton over F. Observe that bool $\mathfrak{S}$ is finite.

5. Left-languages

Let us consider E, F, $\mathfrak{S}$ and let Q be a subset of $\mathfrak{S}$. The functor left-Lang Q yields a subset of E^ω and is defined as follows:

(Def. 4) left-Lang $Q = \{w : Q \text{ meets } w\text{-succ}_{\mathfrak{S}}(\text{the initial state of } \mathfrak{S})\}$.

Next we state the proposition

(16) For every subset Q of $\mathfrak{S}$ holds $w \in$ left-Lang Q iff Q meets w-succ$_{\mathfrak{S}}$(the initial state of $\mathfrak{S}$).

6. Automata

Let us consider E, F. We consider automata over F as extensions of semiautomaton over F as systems

$\langle$ a carrier, a transition, an initial state, final states $\rangle$,

where the carrier is a set, the transition is a relation between the carrier$\times F$ and the carrier, the initial state is a subset of the carrier, and the final states constitute a subset of the carrier.

Let us consider E, F and let $\mathfrak{A}$ be a automaton over F. We say that $\mathfrak{A}$ is deterministic if and only if:

(Def. 5) The semiautomaton of $\mathfrak{A}$ is deterministic.

Let us consider E, F. Observe that there exists a automaton over F which is strict, non empty, finite, and deterministic.

In the sequel $\mathfrak{A}$ denotes a non empty automaton over F and p, q denote elements of $\mathfrak{A}$.

Let us consider E, F, $\mathfrak{A}$. One can check that the transition-system of $\mathfrak{A}$ is non empty and the semiautomaton of $\mathfrak{A}$ is non empty.

Let us consider E, F, $\mathfrak{A}$. The functor $\operatorname{bool}\mathfrak{A}$ yields a strict automaton over $\operatorname{Lex} E$ and is defined by the conditions (Def. 6).

(Def. 6)(i) The semiautomaton of $\operatorname{bool}\mathfrak{A} = \operatorname{bool}$ (the semiautomaton of $\mathfrak{A}$), and

(ii) the final states of $\operatorname{bool}\mathfrak{A} = \{Q; Q$ ranges over elements of $\operatorname{bool}\mathfrak{A} : Q$ meets the final states of $\mathfrak{A}\}$.

Let us consider E, F, $\mathfrak{A}$. One can check that $\operatorname{bool}\mathfrak{A}$ is non empty and deterministic.

The following proposition is true

(17) The carrier of $\operatorname{bool}\mathfrak{A} = 2^{\text{the carrier of } \mathfrak{A}}$.

Let us consider E, F and let $\mathfrak{A}$ be a finite non empty automaton over F. Note that $\operatorname{bool}\mathfrak{A}$ is finite.

7. Right-languages

Let us consider E, F, $\mathfrak{A}$ and let Q be a subset of $\mathfrak{A}$. The functor $\text{right-Lang}\, Q$ yields a subset of E^ω and is defined as follows:

(Def. 7) $\text{right-Lang}\, Q = \{w : w\text{-succ}_{\mathfrak{A}}(Q)$ meets the final states of $\mathfrak{A}\}$.

The following proposition is true

(18) For every subset Q of $\mathfrak{A}$ holds $w \in \text{right-Lang}\, Q$ iff $w\text{-succ}_{\mathfrak{A}}(Q)$ meets the final states of $\mathfrak{A}$.

8. Languages Accepted by Automata

Let us consider E, F, $\mathfrak{A}$. The language generated by $\mathfrak{A}$ yielding a subset of E^ω is defined by the condition (Def. 8).

(Def. 8) The language generated by $\mathfrak{A} = \{u : \bigvee_{p,q} (p \in$ the initial state of $\mathfrak{A} \wedge q \in$ the final states of $\mathfrak{A} \ \wedge \ p, u \Rightarrow^*_{\mathfrak{A}} q)\}$.

The following propositions are true:

(19) $w \in$ the language generated by $\mathfrak{A}$ if and only if there exist p, q such that $p \in$ the initial state of $\mathfrak{A}$ and $q \in$ the final states of $\mathfrak{A}$ and $p, w \Rightarrow^*_{\mathfrak{A}} q$.

(20) $w \in$ the language generated by $\mathfrak{A}$ if and only if w-$\mathrm{succ}_{\mathfrak{A}}$(the initial state of $\mathfrak{A}$) meets the final states of $\mathfrak{A}$.

(21) The language generated by $\mathfrak{A} =$ left-Lang (the final states of $\mathfrak{A}$).

(22) The language generated by $\mathfrak{A} =$ right-Lang (the initial state of $\mathfrak{A}$).

9. Equivalence of Deterministic and Nondeterministic Epsilon Automata

In the sequel $\mathfrak{T}$ denotes a non empty transition-system over $\mathrm{Lex}\, E \cup \{\langle\rangle_E\}$.

One can prove the following three propositions:

(23) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_{\mathbf{2}} = \langle e \rangle \frown u$ and $R(\mathrm{len}\, R)_{\mathbf{2}} = \langle\rangle_E$ holds $R(2)_{\mathbf{2}} = \langle e \rangle \frown u$ or $R(2)_{\mathbf{2}} = u$.

(24) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_{\mathbf{2}} = u$ and $R(\mathrm{len}\, R)_{\mathbf{2}} = \langle\rangle_E$ holds $\mathrm{len}\, R > \mathrm{len}\, u$.

(25) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_{\mathbf{2}} = u \frown v$ and $R(\mathrm{len}\, R)_{\mathbf{2}} = \langle\rangle_E$ there exists l such that $l \in \mathrm{dom}\, R$ and $R(l)_{\mathbf{2}} = v$.

Let us consider E, u, v. The functor $\mathrm{chop}(u, v)$ yielding an element of E^ω is defined by:

(Def. 9)(i) For every w such that $w \frown v = u$ holds $\mathrm{chop}(u, v) = w$ if there exists w such that $w \frown v = u$,

(ii) $\mathrm{chop}(u, v) = u$, otherwise.

The following propositions are true:

(26) Let p be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. Suppose $p(1) = \langle x,\, u \frown w \rangle$ and $p(\mathrm{len}\, p) = \langle y,\, v \frown w \rangle$. Then there exists a reduction sequence q w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $q(1) = \langle x,\, u \rangle$ and $q(\mathrm{len}\, q) = \langle y,\, v \rangle$.

(27) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x,\, u \frown w \rangle$ to $\langle y,\, v \frown w \rangle$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x,\, u \rangle$ to $\langle y,\, v \rangle$.

(28) If $x, u \frown w \Rightarrow^*_{\mathfrak{T}} y, v \frown w$, then $x, u \Rightarrow^*_{\mathfrak{T}} y, v$.

(29) For all elements p, q of $\mathfrak{T}$ such that $p, u \frown v \Rightarrow^*_{\mathfrak{T}} q$ there exists an element r of $\mathfrak{T}$ such that $p, u \Rightarrow^*_{\mathfrak{T}} r$ and $r, v \Rightarrow^*_{\mathfrak{T}} q$.

(30) $w \frown v\text{-succ}_{\mathfrak{T}}(X) = v\text{-succ}_{\mathfrak{T}}(w\text{-succ}_{\mathfrak{T}}(X))$.

(31) $\mathrm{bool}\,\mathfrak{T}$ is a non empty transition-system over $\mathrm{Lex}\,E \cup \{\langle\rangle_E\}$.

(32) $w\text{-succ}_{\mathrm{bool}\,\mathfrak{T}}(\{v\text{-succ}_{\mathfrak{T}}(X)\}) = \{v \frown w\text{-succ}_{\mathfrak{T}}(X)\}$.

In the sequel $\mathfrak{S}$ denotes a non empty semiautomaton over $\mathrm{Lex}\,E \cup \{\langle\rangle_E\}$.

One can prove the following proposition

(33) $w\text{-succ}_{\mathrm{bool}\,\mathfrak{S}}(\{\langle\rangle_E\text{-succ}_{\mathfrak{S}}(X)\}) = \{w\text{-succ}_{\mathfrak{S}}(X)\}$.

In the sequel $\mathfrak{A}$ denotes a non empty automaton over $\mathrm{Lex}\,E \cup \{\langle\rangle_E\}$ and P denotes a subset of $\mathfrak{A}$.

Next we state several propositions:

(34) If $x \in$ the final states of $\mathfrak{A}$ and $x \in P$, then $P \in$ the final states of $\mathrm{bool}\,\mathfrak{A}$.

(35) If $X \in$ the final states of $\mathrm{bool}\,\mathfrak{A}$, then X meets the final states of $\mathfrak{A}$.

(36) The initial state of $\mathrm{bool}\,\mathfrak{A} = \{\langle\rangle_E\text{-succ}_{\mathfrak{A}}(\text{the initial state of }\mathfrak{A})\}$.

(37) $w\text{-succ}_{\mathrm{bool}\,\mathfrak{A}}(\{\langle\rangle_E\text{-succ}_{\mathfrak{A}}(X)\}) = \{w\text{-succ}_{\mathfrak{A}}(X)\}$.

(38) $w\text{-succ}_{\mathrm{bool}\,\mathfrak{A}}(\text{the initial state of }\mathrm{bool}\,\mathfrak{A}) = \{w\text{-succ}_{\mathfrak{A}}(\text{the initial state of }\mathfrak{A})\}$.

(39) The language generated by $\mathfrak{A} =$ the language generated by $\mathrm{bool}\,\mathfrak{A}$.

(40) Let $\mathfrak{A}$ be a non empty automaton over $\mathrm{Lex}\,E \cup \{\langle\rangle_E\}$. Then there exists a non empty deterministic automaton $\mathfrak{A}_1$ over $\mathrm{Lex}\,E$ such that the language generated by $\mathfrak{A} =$ the language generated by $\mathfrak{A}_1$.

(41) Let $\mathfrak{F}$ be a non empty finite automaton over $\mathrm{Lex}\,E \cup \{\langle\rangle_E\}$. Then there exists a non empty deterministic finite automaton $\mathfrak{A}_2$ over $\mathrm{Lex}\,E$ such that the language generated by $\mathfrak{F} =$ the language generated by $\mathfrak{A}_2$.

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