

# Small Inductive Dimension of Topological Spaces

Karol Pałk  
Institute of Computer Science  
University of Białystok  
Poland

**Summary.** We present the concept and basic properties of the Menger-Urysohn small inductive dimension of topological spaces according to the books [7]. Namely, the paper includes the formalization of main theorems from Sections 1.1 and 1.2.

MML identifier: TOPDIM\_1, version: 7.11.04 4.130.1076

The terminology and notation used here are introduced in the following articles: [17], [8], [15], [5], [16], [6], [18], [14], [1], [2], [3], [13], [11], [9], [12], [19], [20], [10], and [4].

## 1. PRELIMINARIES

For simplicity, we adopt the following rules:  $T, T_1, T_2$  denote topological spaces,  $A, B$  denote subsets of  $T$ ,  $F$  denotes a subset of  $T \setminus A$ ,  $G, G_1, G_2$  denote families of subsets of  $T$ ,  $U, W$  denote open subsets of  $T \setminus A$ ,  $p$  denotes a point of  $T \setminus A$ ,  $n$  denotes a natural number, and  $I$  denotes an integer.

One can prove the following propositions:

- (1)  $\text{Fr}(B \cap A) \subseteq \text{Fr} B \cap A$ .
- (2)  $T$  is a  $T_4$  space if and only if for all closed subsets  $A, B$  of  $T$  such that  $A$  misses  $B$  there exist open subsets  $U, W$  of  $T$  such that  $A \subseteq U$  and  $B \subseteq W$  and  $\overline{U}$  misses  $\overline{W}$ .

Let us consider  $T$ . The sequence of  $\text{ind}$  of  $T$  yields a sequence of subsets of  $2^{\text{the carrier of } T}$  and is defined by the conditions (Def. 1).

- (Def. 1)(i) (The sequence of ind of  $T$ )(0) =  $\{\emptyset_T\}$ , and  
(ii)  $A \in$  (the sequence of ind of  $T$ )( $n + 1$ ) iff  $A \in$  (the sequence of ind of  $T$ )( $n$ ) or for all  $p, U$  such that  $p \in U$  there exists  $W$  such that  $p \in W$  and  $W \subseteq U$  and  $\text{Fr } W \in$  (the sequence of ind of  $T$ )( $n$ ).

Let us consider  $T$ . Note that the sequence of ind of  $T$  is ascending.

We now state the proposition

- (3) For every  $F$  such that  $F = B$  holds  $F \in$  (the sequence of ind of  $T \setminus A$ )( $n$ ) iff  $B \in$  (the sequence of ind of  $T$ )( $n$ ).

Let us consider  $T, A$ . We say that  $A$  has finite small inductive dimension if and only if:

- (Def. 2) There exists  $n$  such that  $A \in$  (the sequence of ind of  $T$ )( $n$ ).

Let us consider  $T, A$ . We introduce  $A$  is finite-ind as a synonym of  $A$  has finite small inductive dimension.

Let us consider  $T, G$ . We say that  $G$  has finite small inductive dimension if and only if:

- (Def. 3) There exists  $n$  such that  $G \subseteq$  (the sequence of ind of  $T$ )( $n$ ).

Let us consider  $T, G$ . We introduce  $G$  is finite-ind as a synonym of  $G$  has finite small inductive dimension.

The following proposition is true

- (4) If  $A \in G$  and  $G$  is finite-ind, then  $A$  is finite-ind.

Let us consider  $T$ . One can check the following observations:

- \* every subset of  $T$  which is finite is also finite-ind,
- \* there exists a subset of  $T$  which is finite-ind,
- \* every family of subsets of  $T$  which is empty is also finite-ind, and
- \* there exists a family of subsets of  $T$  which is non empty and finite-ind.

Let  $T$  be a non empty topological space. One can check that there exists a subset of  $T$  which is non empty and finite-ind.

Let us consider  $T$ . We say that  $T$  has finite small inductive dimension if and only if:

- (Def. 4)  $\Omega_T$  has finite small inductive dimension.

Let us consider  $T$ . We introduce  $T$  is finite-ind as a synonym of  $T$  has finite small inductive dimension.

One can verify that every topological space which is empty is also finite-ind.

Let  $X$  be a set. Note that  $\{X\}_{\text{top}}$  is finite-ind.

One can check that there exists a topological space which is non empty and finite-ind.

In the sequel  $A_1$  is a finite-ind subset of  $T$  and  $T_3$  is a finite-ind topological space.

2. SMALL INDUCTIVE DIMENSION

Let us consider  $T$  and let us consider  $A$ . Let us assume that  $A$  is finite-ind. The functor  $\text{ind } A$  yields an integer and is defined as follows:

(Def. 5)  $A \in$  (the sequence of  $\text{ind}$  of  $T$ )( $\text{ind } A + 1$ ) and  $A \notin$  (the sequence of  $\text{ind}$  of  $T$ )( $\text{ind } A$ ).

We now state two propositions:

- (5)  $-1 \leq \text{ind } A_1$ .
- (6)  $\text{ind } A_1 = -1$  iff  $A_1$  is empty.

Let  $T$  be a non empty topological space and let  $A$  be a non empty finite-ind subset of  $T$ . Observe that  $\text{ind } A$  is natural.

The following three propositions are true:

- (7)  $\text{ind } A_1 \leq n - 1$  iff  $A_1 \in$  (the sequence of  $\text{ind}$  of  $T$ )( $n$ ).
- (8) For every finite subset  $A$  of  $T$  holds  $\text{ind } A < \overline{\overline{A}}$ .
- (9)  $\text{ind } A_1 \leq n$  if and only if for every point  $p$  of  $T \setminus A_1$  and for every open subset  $U$  of  $T \setminus A_1$  such that  $p \in U$  there exists an open subset  $W$  of  $T \setminus A_1$  such that  $p \in W$  and  $W \subseteq U$  and  $\text{Fr } W$  is finite-ind and  $\text{ind } \text{Fr } W \leq n - 1$ .

Let us consider  $T$  and let us consider  $G$ . Let us assume that  $G$  is finite-ind. The functor  $\text{ind } G$  yielding an integer is defined by the conditions (Def. 6).

(Def. 6)(i)  $G \subseteq$  (the sequence of  $\text{ind}$  of  $T$ )( $\text{ind } G + 1$ ),  
 (ii)  $-1 \leq \text{ind } G$ , and  
 (iii) for every integer  $i$  such that  $-1 \leq i$  and  $G \subseteq$  (the sequence of  $\text{ind}$  of  $T$ )( $i + 1$ ) holds  $\text{ind } G \leq i$ .

The following propositions are true:

- (10)  $\text{ind } G = -1$  and  $G$  is finite-ind iff  $G \subseteq \{\emptyset_T\}$ .
- (11)  $G$  is finite-ind and  $\text{ind } G \leq I$  iff  $-1 \leq I$  and for every  $A$  such that  $A \in G$  holds  $A$  is finite-ind and  $\text{ind } A \leq I$ .
- (12) If  $G_1$  is finite-ind and  $G_2 \subseteq G_1$ , then  $G_2$  is finite-ind and  $\text{ind } G_2 \leq \text{ind } G_1$ .

Let us consider  $T$  and let  $G_1, G_2$  be finite-ind families of subsets of  $T$ . Observe that  $G_1 \cup G_2$  is finite-ind.

The following proposition is true

- (13) If  $G$  is finite-ind and  $G_1$  is finite-ind and  $\text{ind } G \leq I$  and  $\text{ind } G_1 \leq I$ , then  $\text{ind}(G \cup G_1) \leq I$ .

Let us consider  $T$ . The functor  $\text{ind } T$  yields an integer and is defined as follows:

(Def. 7)  $\text{ind } T = \text{ind}(\Omega_T)$ .

Let  $T$  be a non empty finite-ind topological space. One can verify that  $\text{ind } T$  is natural.

The following three propositions are true:

- (14) For every non empty set  $X$  holds  $\text{ind}(\{X\}_{\text{top}}) = 0$ .
- (15) Given  $n$  such that let  $p$  be a point of  $T$  and  $U$  be an open subset of  $T$ . Suppose  $p \in U$ . Then there exists an open subset  $W$  of  $T$  such that  $p \in W$  and  $W \subseteq U$  and  $\text{Fr } W$  is finite-ind and  $\text{ind Fr } W \leq n - 1$ . Then  $T$  is finite-ind.
- (16)  $\text{ind } T_3 \leq n$  if and only if for every point  $p$  of  $T_3$  and for every open subset  $U$  of  $T_3$  such that  $p \in U$  there exists an open subset  $W$  of  $T_3$  such that  $p \in W$  and  $W \subseteq U$  and  $\text{Fr } W$  is finite-ind and  $\text{ind Fr } W \leq n - 1$ .

### 3. MONOTONICITY OF THE SMALL INDUCTIVE DIMENSION

Let us consider  $T_3$ . Observe that every subset of  $T_3$  is finite-ind.

Let us consider  $T, A_1$ . Note that  $T \setminus A_1$  is finite-ind.

One can prove the following propositions:

- (17)  $\text{ind}(T \setminus A_1) = \text{ind } A_1$ .
- (18) If  $T \setminus A$  is finite-ind, then  $A$  is finite-ind.
- (19) If  $A \subseteq A_1$ , then  $A$  is finite-ind and  $\text{ind } A \leq \text{ind } A_1$ .
- (20) For every subset  $A$  of  $T_3$  holds  $\text{ind } A \leq \text{ind } T_3$ .
- (21) If  $F = B$  and  $B$  is finite-ind, then  $F$  is finite-ind and  $\text{ind } F = \text{ind } B$ .
- (22) If  $F = B$  and  $F$  is finite-ind, then  $B$  is finite-ind and  $\text{ind } F = \text{ind } B$ .
- (23) Let  $T$  be a non empty topological space. Suppose  $T$  is a  $T_3$  space. Then  $T$  is finite-ind and  $\text{ind } T \leq n$  if and only if for every closed subset  $A$  of  $T$  and for every point  $p$  of  $T$  such that  $p \notin A$  there exists a subset  $L$  of  $T$  such that  $L$  separates  $\{p\}, A$  and  $L$  is finite-ind and  $\text{ind } L \leq n - 1$ .
- (24) If  $T_1$  and  $T_2$  are homeomorphic, then  $T_1$  is finite-ind iff  $T_2$  is finite-ind.
- (25) If  $T_1$  and  $T_2$  are homeomorphic and  $T_1$  is finite-ind, then  $\text{ind } T_1 = \text{ind } T_2$ .
- (26) Let  $A_2$  be a subset of  $T_1$  and  $A_3$  be a subset of  $T_2$ . Suppose  $A_2$  and  $A_3$  are homeomorphic. Then  $A_2$  is finite-ind if and only if  $A_3$  is finite-ind.
- (27) Let  $A_2$  be a subset of  $T_1$  and  $A_3$  be a subset of  $T_2$ . If  $A_2$  and  $A_3$  are homeomorphic and  $A_2$  is finite-ind, then  $\text{ind } A_2 = \text{ind } A_3$ .
- (28) If  $T_1 \times T_2$  is finite-ind, then  $T_2 \times T_1$  is finite-ind and  $\text{ind}(T_1 \times T_2) = \text{ind}(T_2 \times T_1)$ .
- (29) For every family  $G_3$  of subsets of  $T \setminus A$  such that  $G_3$  is finite-ind and  $G_3 = G$  holds  $G$  is finite-ind and  $\text{ind } G = \text{ind } G_3$ .
- (30) For every family  $G_3$  of subsets of  $T \setminus A$  such that  $G$  is finite-ind and  $G_3 = G$  holds  $G_3$  is finite-ind and  $\text{ind } G = \text{ind } G_3$ .

## 4. BASIC PROPERTIES 0-DIMENSIONAL TOPOLOGICAL SPACES

Next we state several propositions:

- (31)  $T$  is finite-ind and  $\text{ind } T \leq n$  if and only if there exists a basis  $B_1$  of  $T$  such that for every  $A$  such that  $A \in B_1$  holds  $\text{Fr } A$  is finite-ind and  $\text{ind } \text{Fr } A \leq n - 1$ .
- (32) Let given  $T$ . Suppose that
- (i)  $T$  is a  $T_1$  space, and
  - (ii) for all closed subsets  $A, B$  of  $T$  such that  $A$  misses  $B$  there exist closed subsets  $A', B'$  of  $T$  such that  $A'$  misses  $B'$  and  $A' \cup B' = \Omega_T$  and  $A \subseteq A'$  and  $B \subseteq B'$ .
- Then  $T$  is finite-ind and  $\text{ind } T \leq 0$ .
- (33) Let  $X$  be a set and  $f$  be a sequence of subsets of  $X$ . Then there exists a sequence  $g$  of subsets of  $X$  such that
- (i)  $\bigcup \text{rng } f = \bigcup \text{rng } g$ ,
  - (ii) for all natural numbers  $i, j$  such that  $i \neq j$  holds  $g(i)$  misses  $g(j)$ , and
  - (iii) for every  $n$  there exists a finite family  $f_1$  of subsets of  $X$  such that  $f_1 = \{f(i); i \text{ ranges over elements of } \mathbb{N}: i < n\}$  and  $g(n) = f(n) \setminus \bigcup f_1$ .
- (34) Let given  $T$ . Suppose  $T$  is finite-ind and  $\text{ind } T \leq 0$  and  $T$  is Lindelöf. Let  $A, B$  be closed subsets of  $T$ . Suppose  $A$  misses  $B$ . Then there exist closed subsets  $A', B'$  of  $T$  such that  $A'$  misses  $B'$  and  $A' \cup B' = \Omega_T$  and  $A \subseteq A'$  and  $B \subseteq B'$ .
- (35) Let given  $T$ . Suppose  $T$  is a  $T_1$  space and Lindelöf. Then  $T$  is finite-ind and  $\text{ind } T \leq 0$  if and only if for all closed subsets  $A, B$  of  $T$  such that  $A$  misses  $B$  holds  $\emptyset_T$  separates  $A, B$ .
- (36) Let given  $T$ . Suppose that
- (i)  $T$  is a  $T_4$  space, a  $T_1$  space, and Lindelöf, and
  - (ii) there exists a family  $F$  of subsets of  $T$  such that  $F$  is closed, a cover of  $T$ , countable, and finite-ind and  $\text{ind } F \leq 0$ .
- Then  $T$  is finite-ind and  $\text{ind } T \leq 0$ .

In the sequel  $T_4$  is a metrizable topological space.

We now state four propositions:

- (37) Let  $A, B$  be closed subsets of  $T_4$ . Suppose  $A$  misses  $B$ . Let  $N_1$  be a finite-ind subset of  $T_4$ . Suppose  $\text{ind } N_1 \leq 0$  and  $T_4 \upharpoonright N_1$  is second-countable. Then there exists a subset  $L$  of  $T_4$  such that  $L$  separates  $A, B$  and  $L$  misses  $N_1$ .
- (38) Let  $N_1$  be a subset of  $T_4$ . Suppose  $T_4 \upharpoonright N_1$  is second-countable. Then  $N_1$  is finite-ind and  $\text{ind } N_1 \leq 0$  if and only if for every point  $p$  of  $T_4$  and for every open subset  $U$  of  $T_4$  such that  $p \in U$  there exists an open subset  $W$  of  $T_4$  such that  $p \in W$  and  $W \subseteq U$  and  $N_1$  misses  $\text{Fr } W$ .

- (39) Let  $N_1$  be a subset of  $T_4$ . Suppose  $T_4 \upharpoonright N_1$  is second-countable. Then  $N_1$  is finite-ind and  $\text{ind } N_1 \leq 0$  if and only if there exists a basis  $B$  of  $T_4$  such that for every subset  $A$  of  $T_4$  such that  $A \in B$  holds  $N_1$  misses  $\text{Fr } A$ .
- (40) Let  $N_1, A$  be subsets of  $T_4$ . Suppose  $T_4 \upharpoonright N_1$  is second-countable and  $N_1$  is finite-ind and  $A$  is finite-ind and  $\text{ind } N_1 \leq 0$ . Then  $A \cup N_1$  is finite-ind and  $\text{ind}(A \cup N_1) \leq \text{ind } A + 1$ .

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(4):481–485, 1991.
- [4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [5] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(2):383–386, 1990.
- [6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [7] Roman Duda. *Wprowadzenie do topologii*. PWN, 1986.
- [8] Adam Grabowski. Properties of the product of compact topological spaces. *Formalized Mathematics*, 8(1):55–59, 1999.
- [9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [10] Robert Milewski. Bases of continuous lattices. *Formalized Mathematics*, 7(2):285–294, 1998.
- [11] Andrzej Nędzusiak.  $\sigma$ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [14] Karol Pał. Basic properties of metrizable topological spaces. *Formalized Mathematics*, 17(3):201–205, 2009, doi: 10.2478/v10037-009-0024-8.
- [15] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(2):233–236, 1996.
- [16] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [17] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(4):535–545, 1991.
- [18] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [20] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(1):231–237, 1990.

Received June 29, 2009

---