

Small Inductive Dimension of Topological Spaces. Part II

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Summary. In this paper we present basic properties of n -dimensional topological spaces according to the book [10]. In the article the formalization of Section 1.5 is completed.

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The papers [15], [1], [3], [9], [5], [8], [16], [2], [4], [6], [13], [12], [17], [14], [18], [7], and [11] provide the terminology and notation for this paper.

1. ORDER OF A FAMILY OF SUBSETS OF A SET

In this paper n denotes a natural number, X denotes a set, and F_1, G_1 denote families of subsets of X .

Let us consider X, F_1 . We say that F_1 is finite-order if and only if:

(Def. 1) There exists n such that for every G_1 such that $G_1 \subseteq F_1$ and $n \in \text{Card } G_1$ holds $\bigcap G_1$ is empty.

Let us consider X . Observe that there exists a family of subsets of X which is finite-order and every family of subsets of X which is finite is also finite-order.

Let us consider X, F_1 . The functor order F_1 yielding an extended real number is defined as follows:

(Def. 2)(i) For every G_1 such that order $F_1+1 \in \text{Card } G_1$ and $G_1 \subseteq F_1$ holds $\bigcap G_1$ is empty and there exists G_1 such that $G_1 \subseteq F_1$ but $\text{Card } G_1 = \text{order } F_1+1$ but $\bigcap G_1$ is non empty or G_1 is empty if F_1 is finite-order,
(ii) order $F_1 = +\infty$, otherwise.

Let us consider X and let F be a finite-order family of subsets of X . Observe that order $F + 1$ is natural and order F is integer.

Next we state three propositions:

- (1) If order $F_1 \leq n$, then F_1 is finite-order.
- (2) If order $F_1 \leq n$, then for every G_1 such that $G_1 \subseteq F_1$ and $n+1 \in \text{Card } G_1$ holds $\bigcap G_1$ is empty.
- (3) If for every finite family G of subsets of X such that $G \subseteq F_1$ and $n+1 < \overline{\overline{G}}$ holds $\bigcap G$ is empty, then order $F_1 \leq n$.

2. BASIC PROPERTIES OF n -DIMENSIONAL TOPOLOGICAL SPACES

One can verify that there exists a topological space which is finite-ind, second-countable, and metrizable.

For simplicity, we adopt the following convention: T_1 is a metrizable topological space, T_2, T_3 are finite-ind second-countable metrizable topological spaces, A, B, L, H are subsets of T_1 , U, W are open subsets of T_1 , p is a point of T_1 , F, G are finite families of subsets of T_1 , and I is an integer.

We now state several propositions:

- (4) Let given T_1 . Suppose that
 - (i) T_1 is second-countable, and
 - (ii) there exists F such that F is closed, a cover of T_1 , countable, and finite-ind and $\text{ind } F \leq n$.
 Then T_1 is finite-ind and $\text{ind } T_1 \leq n$.
- (5) Let A, B be finite-ind subsets of T_1 . Suppose A is closed and $T_1 \uparrow (A \cup B)$ is second-countable and $\text{ind } A \leq I$ and $\text{ind } B \leq I$. Then $\text{ind}(A \cup B) \leq I$ and $A \cup B$ is finite-ind.
- (6) Let given T_1 . Suppose T_1 is second-countable and finite-ind and $\text{ind } T_1 \leq n$. Then there exist A, B such that $\Omega_{(T_1)} = A \cup B$ and A misses B and $\text{ind } A \leq n - 1$ and $\text{ind } B \leq 0$.
- (7) Let given T_1 . Suppose T_1 is second-countable and finite-ind and $\text{ind } T_1 \leq I$. Then there exists F such that
 - (i) F is a cover of T_1 and finite-ind,
 - (ii) $\text{ind } F \leq 0$,
 - (iii) $\overline{\overline{F}} \leq I + 1$, and
 - (iv) for all A, B such that $A, B \in F$ and A meets B holds $A = B$.
- (8) Let given T_1 . Suppose T_1 is second-countable and there exists F such that F is a cover of T_1 and finite-ind and $\text{ind } F \leq 0$ and $\overline{\overline{F}} \leq I + 1$. Then T_1 is finite-ind and $\text{ind } T_1 \leq I$.

Let T_1 be a second-countable metrizable topological space and let A, B be finite-ind subsets of T_1 . One can check that $A \cup B$ is finite-ind.

Next we state two propositions:

- (9) If A is finite-ind and B is finite-ind and $T_1 \upharpoonright (A \cup B)$ is second-countable, then $A \cup B$ is finite-ind and $\text{ind}(A \cup B) \leq \text{ind } A + \text{ind } B + 1$.
- (10) For all topological spaces T_4, T_5 and for every subset A_1 of T_4 and for every subset A_2 of T_5 holds $\text{Fr}(A_1 \times A_2) = \text{Fr } A_1 \times \overline{A_2} \cup \overline{A_1} \times \text{Fr } A_2$.

Let us consider T_2, T_3 . Observe that $T_2 \times T_3$ is finite-ind.

We now state several propositions:

- (11) Let given A, B . Suppose A is closed and B is closed and A misses B . Let given H . Suppose $\text{ind } H \leq n$ and $T_1 \upharpoonright H$ is second-countable and finite-ind. Then there exists L such that L separates A, B and $\text{ind}(L \cap H) \leq n - 1$.
- (12) Let given T_1 . Suppose T_1 is finite-ind and second-countable and $\text{ind } T_1 \leq n$. Let given A, B . Suppose A is closed and B is closed and A misses B . Then there exists L such that L separates A, B and $\text{ind } L \leq n - 1$.
- (13) Let given H . Suppose $T_1 \upharpoonright H$ is second-countable. Then H is finite-ind and $\text{ind } H \leq n$ if and only if for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $H \cap \text{Fr } W$ is finite-ind and $\text{ind}(H \cap \text{Fr } W) \leq n - 1$.
- (14) Let given H . Suppose $T_1 \upharpoonright H$ is second-countable. Then H is finite-ind and $\text{ind } H \leq n$ if and only if there exists a basis B_1 of T_1 such that for every A such that $A \in B_1$ holds $H \cap \text{Fr } A$ is finite-ind and $\text{ind}(H \cap \text{Fr } A) \leq n - 1$.
- (15) If T_2 is non empty or T_3 is non empty, then $\text{ind}(T_2 \times T_3) \leq \text{ind } T_2 + \text{ind } T_3$.
- (16) If $\text{ind } T_3 = 0$, then $\text{ind}(T_2 \times T_3) = \text{ind } T_2$.

3. SMALL INDUCTIVE DIMENSION OF EUCLIDEAN SPACES

For simplicity, we follow the rules: u denotes a point of \mathcal{E}^1 , U denotes a point of \mathcal{E}_T^1 , r, u_1 denote real numbers, and s denotes a real number.

Next we state three propositions:

- (17) If $\langle u_1 \rangle = u$ and $r > 0$, then $\overline{\text{Ball}}(u, r) = \{ \langle s \rangle : u_1 - r \leq s \wedge s \leq u_1 + r \}$.
- (18) If $\langle u_1 \rangle = U$ and $r > 0$, then $\text{Fr Ball}(U, r) = \{ \langle u_1 - r \rangle, \langle u_1 + r \rangle \}$.
- (19) Let T be a topological space and A be a countable subset of T . If $T \upharpoonright A$ is a T_4 space, then A is finite-ind and $\text{ind } A \leq 0$.

Let T_1 be a metrizable topological space. Observe that every subset of T_1 which is countable is also finite-ind.

Let n be a natural number. Observe that \mathcal{E}_T^n is finite-ind.

One can prove the following propositions:

- (20) If $n \leq 1$, then $\text{ind}(\mathcal{E}_T^n) = n$.
- (21) $\text{ind}(\mathcal{E}_T^n) \leq n$.

- (22) Let given A . Suppose $T_1 \upharpoonright A$ is second-countable and finite-ind and $\text{ind } A \leq 0$. Let given F . Suppose F is open and a cover of A . Then there exists a function g from F into $2^{\text{the carrier of } T_1}$ such that
- (i) $\text{rng } g$ is open,
 - (ii) $\text{rng } g$ is a cover of A ,
 - (iii) for every set a such that $a \in F$ holds $g(a) \subseteq a$, and
 - (iv) for all sets a, b such that $a, b \in F$ and $a \neq b$ holds $g(a)$ misses $g(b)$.
- (23) Let given T_1 . Suppose T_1 is second-countable and finite-ind and $\text{ind } T_1 \leq n$. Let given F . Suppose F is open and a cover of T_1 . Then there exists G such that G is open, a cover of T_1 , and finer than F and $\overline{G} \leq \overline{F} \cdot (n + 1)$ and order $G \leq n$.
- (24) Let given T_1 . Suppose T_1 is finite-ind. Let given A . Suppose $\text{ind}(A^c) \leq n$ and $T_1 \upharpoonright A^c$ is second-countable. Let A_1, A_2 be closed subsets of T_1 . Suppose $A = A_1 \cup A_2$. Then there exist closed subsets X_1, X_2 of T_1 such that $\Omega_{(T_1)} = X_1 \cup X_2$ and $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ and $A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$ and $\text{ind}(X_1 \cap X_2 \setminus A_1 \cap A_2) \leq n - 1$.

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