Dilworth’s Decomposition Theorem for Posets

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Summary. The following theorem is due to Dilworth [8]: Let $P$ be a partially ordered set. If the maximal number of elements in an independent subset (anti-chain) of $P$ is $k$, then $P$ is the union of $k$ chains (cliques).

In this article we formalize an elegant proof of the above theorem for finite posets by Perles [13]. The result is then used in proving the case of infinite posets following the original proof of Dilworth [8].

A dual of Dilworth’s theorem also holds: a poset with maximum clique $m$ is a union of $m$ independent sets. The proof of this dual fact is considerably easier; we follow the proof by Mirsky [11]. Mirsky states also a corollary that a poset of $r \times s + 1$ elements possesses a clique of size $r + 1$ or an independent set of size $s + 1$, or both. This corollary is then used to prove the result of Erdős and Szekeres [9].

Instead of using posets, we drop reflexivity and state the facts about antisymmetric and transitive relations.

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The articles [1], [15], [14], [7], [2], [16], [3], [12], [17], [5], [10], [4], and [6] provide the notation and terminology for this paper.

1. Preliminaries

The scheme FraenkelFinCard1 deals with a finite non empty set $A$, a finite set $B$, a unary functor $F$ yielding a set, and a unary predicate $P$, and states that:

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provided the following condition is satisfied:

- \( B = \{ F(w); w \text{ ranges over elements of } A : P[w] \} \).

Next we state the proposition

(1) For all sets \( X, Y, x \) such that \( x \notin X \) holds \( X \setminus (Y \cup \{ x \}) = X \setminus Y \).

Let us note that every set which is empty is also \( \subseteq \)-linear and there exists a set which is empty and \( \subseteq \)-linear.

Let \( X \) be a \( \subseteq \)-linear set. Note that every subset of \( X \) is \( \subseteq \)-linear.

One can prove the following four propositions:

(2) Let \( X, Y \) be sets, \( F \) be a family of subsets of \( X \), and \( G \) be a family of subsets of \( Y \). Then \( F \cup G \) is a family of subsets of \( X \cup Y \).

(3) Let \( X, Y \) be sets, \( F \) be a partition of \( X \), and \( G \) be a partition of \( Y \). If \( X \) misses \( Y \), then \( F \cup G \) is a partition of \( X \cup Y \).

(4) For all sets \( X, Y \) and for every partition \( F \) of \( Y \) such that \( Y \subset X \) holds \( F \cup \{ X \setminus Y \} \) is a partition of \( X \).

(5) For every infinite set \( X \) and for every natural number \( n \) there exists a finite subset \( Y \) of \( X \) such that \( Y > n \).

## 2.Cliques and Stable Sets

Let \( R \) be a relational structure and let \( S \) be a subset of \( R \). We say that \( S \) is connected if and only if:

(Def. 1) The internal relation of \( R \) is connected in \( S \).

Let \( R \) be a relational structure and let \( S \) be a subset of \( R \). We introduce \( S \) is a clique as a synonym of \( S \) is connected.

Let \( R \) be a relational structure. Note that every subset of \( R \) which is trivial is also a clique.

Let \( R \) be a relational structure. One can check that there exists a subset of \( R \) which is a clique.

Let \( R \) be a relational structure. A clique of \( R \) is a clique subset of \( R \).

We now state the proposition

(6) Let \( R \) be a relational structure and \( S \) be a subset of \( R \). Then \( S \) is a clique of \( R \) if and only if for all elements \( a, b \) of \( R \) such that \( a, b \in S \) and \( a \neq b \) holds \( a \leq b \) or \( b \leq a \).

Let \( R \) be a relational structure. Observe that there exists a clique of \( R \) which is finite.

Let \( R \) be a reflexive relational structure. One can check that every subset of \( R \) which is connected is also strongly connected.

Let \( R \) be a non empty relational structure. Observe that there exists a clique of \( R \) which is finite and non empty.
One can prove the following propositions:

(7) Let $R$ be a non empty relational structure and $a_1, a_2$ be elements of $R$. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a clique of $R$, then $a_1 \leq a_2$ or $a_2 \leq a_1$.

(8) Let $R$ be a non empty relational structure and $a_1, a_2$ be elements of $R$. If $a_1 \leq a_2$ or $a_2 \leq a_1$, then $\{a_1, a_2\}$ is a clique of $R$.

(9) For every relational structure $R$ and for every clique $C$ of $R$ holds every subset of $C$ is a clique of $R$.

(10) Let $R$ be a relational structure, $C$ be a finite clique of $R$, and $n$ be a natural number. If $n \leq C$, then there exists a finite clique $B$ of $R$ such that $B = n$.

(11) Let $R$ be a transitive relational structure, $C$ be a clique of $R$, and $x, y$ be elements of $R$. If $x$ is maximal in $C$ and $x \leq y$, then $C \cup \{y\}$ is a clique of $R$.

(12) Let $R$ be a transitive relational structure, $C$ be a clique of $R$, and $x, y$ be elements of $R$. If $x$ is minimal in $C$ and $y \leq x$, then $C \cup \{y\}$ is a clique of $R$.

Let $R$ be a relational structure and let $S$ be a subset of $R$. We say that $S$ is stable if and only if:

(Def. 2) For all elements $x, y$ of $R$ such that $x, y \in S$ and $x \neq y$ holds $x \not\leq y$ and $y \not\leq x$.

Let $R$ be a relational structure. One can check that every subset of $R$ which is trivial is also stable. Let $R$ be a relational structure. Note that there exists a subset of $R$ which is stable.

Let $R$ be a relational structure. A stable set of $R$ is a stable subset of $R$.

Let $R$ be a relational structure. Note that there exists a stable set of $R$ which is finite.

Let $R$ be a non empty relational structure. Observe that there exists a stable set of $R$ which is finite and non empty.

The following propositions are true:

(13) Let $R$ be a non empty relational structure and $a_1, a_2$ be elements of $R$. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a stable set of $R$, then $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$.

(14) Let $R$ be a non empty relational structure and $a_1, a_2$ be elements of $R$. If $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, then $\{a_1, a_2\}$ is a stable set of $R$.

(15) Let $R$ be a relational structure, $C$ be a clique of $R$, $A$ be a stable set of $R$, and $a, b$ be sets. If $a, b \in A$ and $a, b \in C$, then $a = b$.

(16) For every relational structure $R$ and for every stable set $A$ of $R$ holds every subset of $A$ is a stable set of $R$.

(17) Let $R$ be a relational structure, $A$ be a finite stable set of $R$, and $n$ be a natural number. If $n \leq \overline{A}$, then there exists a finite stable set $B$ of $R$ such that $\overline{B} = n$. 
3. CLIQUE NUMBER AND STABILITY NUMBER

Let $R$ be a relational structure. We say that $R$ has finite clique number if and only if:

(Def. 3) There exists a finite clique $C$ of $R$ such that for every finite clique $D$ of $R$ holds $\overline{D} \leq \overline{C}$.

Let us observe that every relational structure which is finite has also finite clique number and there exists a relational structure which is non empty, antisymmetric, and transitive and has finite clique number.

Let $R$ be a relational structure with finite clique number. Observe that every clique of $R$ is finite.

Let $R$ be a relational structure with finite clique number. The functor $\omega(R)$ yields a natural number and is defined as follows:

(Def. 4) There exists a finite clique $C$ of $R$ such that $\overline{C} = \omega(R)$ and for every finite clique $T$ of $R$ holds $\overline{T} \leq \omega(R)$.

Let $R$ be an empty relational structure. Note that $\omega(R)$ is empty.

Let $R$ be a non empty relational structure with finite clique number. Observe that $\omega(R)$ is positive.

Next we state two propositions:

(18) For every non empty relational structure $R$ with finite clique number such that $\Omega_R$ is a stable set of $R$ holds $\omega(R) = 1$.

(19) For every relational structure $R$ with finite clique number such that $\omega(R) = 1$ holds $\Omega_R$ is a stable set of $R$.

Let $R$ be a relational structure. We say that $R$ has finite stability number if and only if:

(Def. 5) There exists a finite stable set $A$ of $R$ such that for every finite stable set $B$ of $R$ holds $\overline{B} \leq \overline{A}$.

One can verify that every relational structure which is finite has also finite stability number and there exists a relational structure which is antisymmetric, transitive, and non empty and has finite stability number.

Let $R$ be a relational structure with finite stability number. Note that every stable set of $R$ is finite.

Let $R$ be a relational structure with finite stability number. The functor $\alpha(R)$ yielding a natural number is defined by:

(Def. 6) There exists a finite stable set $A$ of $R$ such that $\overline{A} = \alpha(R)$ and for every finite stable set $T$ of $R$ holds $\overline{T} \leq \alpha(R)$.

Let $R$ be an empty relational structure. Observe that $\alpha(R)$ is empty.

Let $R$ be a non empty relational structure with finite stability number. One can verify that $\alpha(R)$ is positive.

We now state two propositions:
(20) For every non empty relational structure $R$ with finite stability number such that $\Omega_R$ is a clique of $R$ holds $\alpha(R) = 1$.

(21) For every relational structure $R$ with finite stability number such that $\alpha(R) = 1$ holds $\Omega_R$ is a clique of $R$.

Let us mention that every relational structure which has finite clique number and finite stability number is also finite.

4. LOWER AND UPPER SETS, MINIMAL AND MAXIMAL ELEMENTS

Let $R$ be a relational structure and let $X$ be a subset of $R$. The functor $\text{Lower } X$ yields a subset of $R$ and is defined by:

(Def. 7) $\text{Lower } X = X \cup \downarrow X$.

The functor $\text{Upper } X$ yielding a subset of $R$ is defined as follows:

(Def. 8) $\text{Upper } X = X \cup \uparrow X$.

One can prove the following propositions:

(22) Let $R$ be an antisymmetric transitive relational structure, $A$ be a stable set of $R$, and $z$ be a set. If $z \in \text{Upper } A$ and $z \in \text{Lower } A$, then $z \in A$.

(23) Let $R$ be a relational structure with finite stability number and $A$ be a stable set of $R$. If $\overline{A} = \alpha(R)$, then $\text{Upper } A \cup \text{Lower } A = \Omega_R$.

(24) Let $R$ be a transitive relational structure, $x$ be an element of $R$, and $S$ be a subset of $R$. If $x$ is minimal in $\text{Lower } S$, then $x$ is minimal in $\Omega_R$.

(25) Let $R$ be a transitive relational structure, $x$ be an element of $R$, and $S$ be a subset of $R$. If $x$ is maximal in $\text{Upper } S$, then $x$ is maximal in $\Omega_R$.

Let $R$ be a relational structure. The functor $\text{minimals}(R)$ yielding a subset of $R$ is defined as follows:

(Def. 9)(i) For every element $x$ of $R$ holds $x \in \text{minimals}(R)$ iff $x$ is minimal in $\Omega_R$ if $R$ is non empty,

(ii) $\text{minimals}(R) = \emptyset$, otherwise.

The functor $\text{maximals}(R)$ yielding a subset of $R$ is defined as follows:

(Def. 10)(i) For every element $x$ of $R$ holds $x \in \text{maximals}(R)$ iff $x$ is maximal in $\Omega_R$ if $R$ is non empty,

(ii) $\text{maximals}(R) = \emptyset$, otherwise.

Let $R$ be a non empty antisymmetric transitive relational structure with finite clique number. One can verify that $\text{maximals}(R)$ is non empty and $\text{minimals}(R)$ is non empty.

Let $R$ be a relational structure. Note that $\text{minimals}(R)$ is stable and $\text{maximals}(R)$ is stable.

The following two propositions are true:
For every relational structure $R$ and for every stable set $A$ of $R$ such that $\text{minimals}(R) \not\subseteq A$ holds $\text{minimals}(R) \not\subseteq \text{Upper } A$.

For every relational structure $R$ and for every stable set $A$ of $R$ such that $\text{maximals}(R) \not\subseteq A$ holds $\text{maximals}(R) \not\subseteq \text{Lower } A$.

5. Substructures

Let $R$ be a relational structure and let $X$ be a finite subset of $R$. Observe that $\text{sub}(X)$ is finite.

One can prove the following propositions:

(28) For every relational structure $R$ and for every subset $S$ of $R$ holds every clique of $\text{sub}(S)$ is a clique of $R$.

(29) Let $R$ be a relational structure, $S$ be a subset of $R$, and $C$ be a clique of $R$. Then $C \cap S$ is a clique of $\text{sub}(S)$.

(30) For every relational structure $R$ and for every subset $S$ of $R$ holds every stable set of $\text{sub}(S)$ is a stable set of $R$.

(31) Let $R$ be a relational structure, $S$ be a subset of $R$, and $A$ be a stable set of $R$. Then $A \cap S$ is a stable set of $\text{sub}(S)$.

(32) Let $R$ be a relational structure, $S$ be a subset of $R$, $B$ be a subset of $\text{sub}(S)$, $x$ be an element of $\text{sub}(S)$, and $y$ be an element of $R$. If $x = y$ and $x$ is maximal in $B$, then $y$ is maximal in $B$.

(33) Let $R$ be a relational structure, $S$ be a subset of $R$, $B$ be a subset of $\text{sub}(S)$, $x$ be an element of $\text{sub}(S)$, and $y$ be an element of $R$. If $x = y$ and $x$ is minimal in $B$, then $y$ is minimal in $B$.

(34) Let $R$ be a transitive relational structure, $A$ be a stable set of $R$, $C$ be a clique of $\text{sub}(\text{Lower } A)$, and $a, b$ be elements of $R$. If $a \in A$ and $a, b \in C$, then $a = b$ or $b \leq a$.

(35) Let $R$ be a transitive relational structure, $A$ be a stable set of $R$, $C$ be a clique of $\text{sub}(\text{Upper } A)$, and $a, b$ be elements of $R$. If $a \in A$ and $a, b \in C$, then $a = b$ or $b \leq a$.

Let $R$ be a relational structure with finite clique number and let $S$ be a subset of $R$. One can verify that $\text{sub}(S)$ has finite clique number.

Let $R$ be a relational structure with finite stability number and let $S$ be a subset of $R$. One can verify that $\text{sub}(S)$ has finite stability number.

The following propositions are true:

(36) Let $R$ be a non empty antisymmetric transitive relational structure with finite clique number and $x$ be an element of $R$. Then there exists an element $y$ of $R$ such that $y$ is minimal in $\Omega_R$ but $y = x$ or $y < x$.

(37) For every antisymmetric transitive relational structure $R$ with finite clique number holds $\text{Upper minimals}(R) = \Omega_R$. 
(38) Let $R$ be a non-empty antisymmetric transitive relational structure with finite clique number and $x$ be an element of $R$. Then there exists an element $y$ of $R$ such that $y$ is maximal in $\Omega_R$ but $y = x$ or $x < y$.

(39) For every antisymmetric transitive relational structure $R$ with finite clique number holds $\text{Lower maximals}(R) = \Omega_R$.

(40) Let $R$ be an antisymmetric transitive relational structure with finite clique number and $A$ be a stable set of $R$. If $\text{minimals}(R) \subseteq A$, then $A = \text{minimals}(R)$.

(41) Let $R$ be an antisymmetric transitive relational structure with finite clique number and $A$ be a stable set of $R$. If $\text{maximals}(R) \subseteq A$, then $A = \text{maximals}(R)$.

(42) For every relational structure $R$ with finite clique number and for every subset $S$ of $R$ holds $\omega(\text{sub}(S)) \leq \omega(R)$.

(43) Let $R$ be a relational structure with finite clique number, $C$ be a clique of $R$, and $S$ be a subset of $R$. If $\overline{C} = \omega(R)$ and $C \subseteq S$, then $\omega(\text{sub}(S)) = \omega(R)$.

(44) For every relational structure $R$ with finite stability number and for every subset $S$ of $R$ holds $\alpha(\text{sub}(S)) \leq \alpha(R)$.

(45) Let $R$ be a relational structure with finite stability number, $A$ be a stable set of $R$, and $S$ be a subset of $R$. If $\overline{A} = \alpha(R)$ and $A \subseteq S$, then $\alpha(\text{sub}(S)) = \alpha(R)$.

6. Partitions into Cliques and Stable Sets

Let $R$ be a relational structure and let $P$ be a partition of the carrier of $R$. We say that $P$ is clique-wise if and only if:

(Def. 11) For every set $x$ such that $x \in P$ holds $x$ is a clique of $R$.

Let $R$ be a relational structure. Observe that there exists a partition of the carrier of $R$ which is clique-wise.

Let $R$ be a relational structure. A clique-partition of $R$ is a clique-wise partition of the carrier of $R$.

Let $R$ be an empty relational structure. One can verify that every partition of the carrier of $R$ which is empty is also clique-wise.

Next we state four propositions:

(46) For every finite relational structure $R$ and for every clique-partition $C$ of $R$ holds $\overline{C} \geq \alpha(R)$.

(47) Let $R$ be a relational structure with finite stability number, $A$ be a stable set of $R$, and $C$ be a clique-partition of $R$. Suppose $\text{Card} C = \text{Card} A$. Then there exists a function $f$ from $A$ into $C$ such that $f$ is bijective and for every set $x$ such that $x \in A$ holds $x \in f(x)$.
Let $R$ be a finite relational structure, $A$ be a stable set of $R$, and $C$ be a clique-partition of $R$. Suppose $\overline{C} = \overline{A}$. Let $c$ be a set. If $c \in C$, then there exists an element $a$ of $A$ such that $c \cap A = \{a\}$.

Let $R$ be an antisymmetric transitive non empty relational structure with finite stability number, $A$ be a stable set of $R$, $U$ be a clique-partition of sub(Upper $A$), and $L$ be a clique-partition of sub(Lower $A$). Suppose $\overline{A} = \alpha(R)$ and Card $U = \alpha(R)$ and Card $L = \alpha(R)$. Then there exists a clique-partition $C$ of $R$ such that Card $C = \alpha(R)$.

Let $R$ be a relational structure and let $P$ be a partition of the carrier of $R$. We say that $P$ is stable-wise if and only if:

(Def. 12) For every set $x$ such that $x \in P$ holds $x$ is a stable set of $R$.

Let $R$ be a relational structure. Observe that there exists a partition of the carrier of $R$ which is stable-wise.

Let $R$ be a relational structure. A coloring of $R$ is a stable-wise partition of the carrier of $R$.

Let $R$ be an empty relational structure. Note that every partition of the carrier of $R$ is stable-wise.

We now state the proposition

(50) For every finite relational structure $R$ and for every coloring $C$ of $R$ holds $\overline{C} \geq \omega(R)$.

7. Dilworth’s Theorem and a Dual

Next we state the proposition

(51) Let $R$ be a finite antisymmetric transitive relational structure. Then there exists a clique-partition $C$ of $R$ such that $\overline{C} = \alpha(R)$.

Let $R$ be a non empty relational structure with finite stability number and let $C$ be a subset of $R$. We say that $C$ is strong-chain if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let $S$ be a finite non empty subset of $R$. Then there exists a clique-partition $P$ of sub($S$) such that $\overline{P} \leq \alpha(R)$ and there exists a set $c$ such that $c \in P$ and $S \cap C \subseteq c$ and for every set $d$ such that $d \in P$ and $d \neq c$ holds $C \cap d = \emptyset$.

Let $R$ be a non empty relational structure with finite stability number. Note that every subset of $R$ which is strong-chain is also a clique.

Let $R$ be an antisymmetric transitive non empty relational structure with finite stability number. Observe that every subset of $R$ which is trivial and non empty is also strong-chain.

The following propositions are true:
(52) Let $R$ be a non empty antisymmetric transitive relational structure with finite stability number. Then there exists a non empty subset $S$ of $R$ such that $S$ is strong-chain and it is not true that there exists a subset $D$ of $R$ such that $D$ is strong-chain and $S \subset D$.

(53) Let $R$ be an antisymmetric transitive relational structure with finite stability number. Then there exists a clique-partition $C$ of $R$ such that $\text{Card } C = \alpha(R)$.

(54) Let $R$ be an antisymmetric transitive relational structure with finite clique number. Then there exists a coloring $A$ of $R$ such that $\text{Card } A = \omega(R)$.

8. Erdős-Szekeres Theorem

One can prove the following two propositions:

(55) Let $R$ be a finite antisymmetric transitive relational structure and $r$, $s$ be natural numbers. Suppose $\text{Card } R = r \cdot s + 1$. Then there exists a clique $C$ of $R$ such that $\overline{C} \geq r + 1$ or there exists a stable set $A$ of $R$ such that $\overline{A} \geq s + 1$.

(56) Let $f$ be a real-valued finite sequence and $n$ be a natural number. Suppose $\overline{f} = n^2 + 1$ and $f$ is one-to-one. Then there exists a real-valued finite subsequence $g$ such that $g \subseteq f$ and $\overline{g} \geq n + 1$ and $g$ is increasing or decreasing.

References


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