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Vector Functions and their Differentiation Formulas in 3-dimensional Euclidean Spaces

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Summary. In this article, we first extend several basic theorems of the operation of vector in 3-dimensional Euclidean spaces. Then three unit vectors: $e1$, $e2$, $e3$ and the definition of vector function in the same spaces are introduced. By dint of unit vector the main operation properties as well as the differentiation formulas of vector function are shown [12].

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The notation and terminology used in this paper have been introduced in the following papers: [7], [11], [2], [3], [4], [1], [5], [8], [9], [6], [10], and [13].

1. Preliminaries

For simplicity, we use the following convention: r, r_1, r_2, x, y, z, x_1, x_2, x_3, y_1, y_2, y_3 are elements of $\mathbb{R}$, p, q, p_1, p_2, p_3, q_1, q_2 are elements of $\mathcal{R}^3$, f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and t, t_0, t_1, t_2 are real numbers.

Let x, y, z be real numbers. Then $[x, y, z]$ is an element of $\mathcal{R}^3$.

One can prove the following proposition

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(1) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = 3$ holds f is an element of $\mathcal{R}^3$.

The element e_1 of $\mathcal{R}^3$ is defined by:

(Def. 1) $e_1 = [1, 0, 0]$.

The element e_2 of $\mathcal{R}^3$ is defined as follows:

(Def. 2) $e_2 = [0, 1, 0]$.

The element e_3 of $\mathcal{R}^3$ is defined as follows:

(Def. 3) $e_3 = [0, 0, 1]$.

Let us consider p_1, p_2. The functor $p_1 \times p_2$ yielding an element of $\mathcal{R}^3$ is defined as follows:

(Def. 4) $p_1 \times p_2 = [p_1(2) \cdot p_2(3) - p_1(3) \cdot p_2(2), p_1(3) \cdot p_2(1) - p_1(1) \cdot p_2(3), p_1(1) \cdot p_2(2) - p_1(2) \cdot p_2(1)]$.

Next we state the proposition

(2) If p_1 and p_2 are linearly dependent, then $p_1 \times p_2 = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

2. Vector Functions in 3-dimensional Euclidean Spaces

We now state a number of propositions:

(3) $|e_1| = 1$.

(4) $|e_2| = 1$.

(5) $|e_3| = 1$.

(6) e_1, e_2 are orthogonal.

(7) e_1, e_3 are orthogonal.

(8) e_2, e_3 are orthogonal.

(9) $|(e_1, e_1)| = 1$.

(10) $|(e_2, e_2)| = 1$.

(11) $|(e_3, e_3)| = 1$.

(12) $|(e_1, [0, 0, 0])| = 0$.

(13) $|(e_2, [0, 0, 0])| = 0$.

(14) $|(e_3, [0, 0, 0])| = 0$.

(15) $e_1 \times e_2 = e_3$.

(16) $e_2 \times e_3 = e_1$.

(17) $e_3 \times e_1 = e_2$.

(18) $e_3 \times e_2 = -e_1$.

(19) $e_1 \times e_3 = -e_2$.

(20) $e_2 \times e_1 = -e_3$.

(21) $e_1 \times [0, 0, 0] = [0, 0, 0]$.

(22) $e_2 \times [0, 0, 0] = [0, 0, 0]$.

(23) $e_3 \times [0, 0, 0] = [0, 0, 0]$.

(24) $r \cdot e_1 = [r, 0, 0]$.

(25) $r \cdot e_2 = [0, r, 0]$.

(26) $r \cdot e_3 = [0, 0, r]$.

(27) $1 \cdot e_1 = e_1$.

(28) $1 \cdot e_2 = e_2$.

(29) $1 \cdot e_3 = e_3$.

(30) $-e_1 = [-1, 0, 0]$.

(31) $-e_2 = [0, -1, 0]$.

(32) $-e_3 = [0, 0, -1]$.

(33) $0 \cdot e_1 = [0, 0, 0]$.

(34) $0 \cdot e_2 = [0, 0, 0]$.

(35) $0 \cdot e_3 = [0, 0, 0]$.

(36) $p = p(1) \cdot e_1 + p(2) \cdot e_2 + p(3) \cdot e_3$.

(37) $r \cdot p = r \cdot p(1) \cdot e_1 + r \cdot p(2) \cdot e_2 + r \cdot p(3) \cdot e_3$.

(38) $[x, y, z] = x \cdot e_1 + y \cdot e_2 + z \cdot e_3$.

(39) $r \cdot [x, y, z] = r \cdot x \cdot e_1 + r \cdot y \cdot e_2 + r \cdot z \cdot e_3$.

(40) $-p = -p(1) \cdot e_1 - p(2) \cdot e_2 - p(3) \cdot e_3$.

(41) $-[x, y, z] = -x \cdot e_1 - y \cdot e_2 - z \cdot e_3$.

(42) $p_1 + p_2 = (p_1(1) + p_2(1)) \cdot e_1 + (p_1(2) + p_2(2)) \cdot e_2 + (p_1(3) + p_2(3)) \cdot e_3$.

(43) $p_1 - p_2 = (p_1(1) - p_2(1)) \cdot e_1 + (p_1(2) - p_2(2)) \cdot e_2 + (p_1(3) - p_2(3)) \cdot e_3$.

(44) $[x_1, x_2, x_3] + [y_1, y_2, y_3] = (x_1 + y_1) \cdot e_1 + (x_2 + y_2) \cdot e_2 + (x_3 + y_3) \cdot e_3$.

(45) $[x_1, x_2, x_3] - [y_1, y_2, y_3] = (x_1 - y_1) \cdot e_1 + (x_2 - y_2) \cdot e_2 + (x_3 - y_3) \cdot e_3$.

(46) $p_1(1) \cdot e_1 + p_1(2) \cdot e_2 + p_1(3) \cdot e_3 = (p_2(1) + p_3(1)) \cdot e_1 + (p_2(2) + p_3(2)) \cdot e_2 + (p_2(3) + p_3(3)) \cdot e_3$ if and only if $p_2(1) \cdot e_1 + p_2(2) \cdot e_2 + p_2(3) \cdot e_3 = (p_1(1) - p_3(1)) \cdot e_1 + (p_1(2) - p_3(2)) \cdot e_2 + (p_1(3) - p_3(3)) \cdot e_3$.

Let f_1, f_2, f_3 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\mathrm{VFunc}(f_1, f_2, f_3)$ yielding a function from $\mathbb{R}$ into $\mathcal{R}^3$ is defined as follows:

(Def. 5) For every t holds $(\mathrm{VFunc}(f_1, f_2, f_3))(t) = [f_1(t), f_2(t), f_3(t)]$.

We now state a number of propositions:

(47) $(\mathrm{VFunc}(f_1, f_2, f_3))(t) = f_1(t) \cdot e_1 + f_2(t) \cdot e_2 + f_3(t) \cdot e_3$.

(48) $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ iff $p(1) = f_1(t)$ and $p(2) = f_2(t)$ and $p(3) = f_3(t)$.

(49) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}\, p = 3$ and $\mathrm{dom}\, p = \mathrm{Seg}\, 3$.

(50) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p \bullet q = \langle f_1(t_1) \cdot g_1(t_2), f_2(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_3(t_2) \rangle$.

(51) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r \cdot p = [r \cdot f_1(t), r \cdot f_2(t), r \cdot f_3(t)]$.

(52) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $-p = [-f_1(t), -f_2(t), -f_3(t)]$.

(53) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $(-p)(1) = -f_1(t)$ and $(-p)(2) = -f_2(t)$ and $(-p)(3) = -f_3(t)$.

(54) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}(-p) = 3$.

(55) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}(-p) = \mathrm{len}\, p$.

(56) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $\mathrm{len}(p + q) = 3$.

(57) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + q = [f_1(t_1) + g_1(t_2), f_2(t_1) + g_2(t_2), f_3(t_1) + g_3(t_2)]$.

(58) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$ and $p = q$, then $f_1(t_1) = g_1(t_2)$ and $f_2(t_1) = g_2(t_2)$ and $f_3(t_1) = g_3(t_2)$.

(59) If $f_1(t_1) = g_1(t_2)$ and $f_2(t_1) = g_2(t_2)$ and $f_3(t_1) = g_3(t_2)$, then $(\mathrm{VFunc}(f_1, f_2, f_3))(t_1) = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$.

(60) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + q = [f_1(t_1) + g_1(t_2), f_2(t_1) + g_2(t_2), f_3(t_1) + g_3(t_2)]$.

(61) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + -q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)]$.

(62) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p - q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)]$.

(63) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $\mathrm{len}(p - q) = 3$.

(64) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p, q)| = f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2)$.

(65) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|(p, p)| = f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}$.

(66) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|p| = \sqrt{f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}}$.

(67) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|r \cdot p| = |r| \cdot \sqrt{f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}}$.

(68) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p \times q = [f_2(t_1) \cdot g_3(t_2) - f_3(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_1(t_2) - f_1(t_1) \cdot g_3(t_2), f_1(t_1) \cdot g_2(t_2) - f_2(t_1) \cdot g_1(t_2)]$.

(69) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r_1 \cdot p + r_2 \cdot p = (r_1 + r_2) \cdot [f_1(t), f_2(t), f_3(t)]$.

(70) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r_1 \cdot p - r_2 \cdot p = (r_1 - r_2) \cdot [f_1(t), f_2(t), f_3(t)]$.

(71) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(r \cdot p, q)| = r \cdot (f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2))$.

(72) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|(p, 0_{\mathcal{E}^3_{\mathrm{T}}})| = 0$.

(73) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(-p, q)| = -|(p, q)|$.

(74) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(-p, -q)| = |(p, q)|$.

(75) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 - p_2, q)| = |(p_1, q)| - |(p_2, q)|$.

(76) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 + p_2, q)| = |(p_1, q)| + |(p_2, q)|$.

(77) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(r_1 \cdot p_1 + r_2 \cdot p_2, q)| = r_1 \cdot |(p_1, q)| + r_2 \cdot |(p_2, q)|$.

(78) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q_1 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_1)$ and $q_2 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_1, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|$.

(79) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q_1 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_1)$ and $q_2 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 - p_2, q_1 - q_2)| = (|(p_1, q_1)| - |(p_1, q_2)| - |(p_2, q_1)|) + |(p_2, q_2)|$.

(80) For every p such that $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ holds $|(p, p)| = 0$ iff $p = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(81) For every p such that $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ holds $|p| = 0$ iff $p = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(82) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $|(p - q, p - q)| = (|(p, p)| - 2 \cdot |(p, q)|) + |(q, q)|$.

(83) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $|(p + q, p + q)| = |(p, p)| + 2 \cdot |(p, q)| + |(q, q)|$.

(84) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $(r \cdot p) \times q = r \cdot (p \times q)$ and $(r \cdot p) \times q = p \times (r \cdot q)$.

(85) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $p_1 \times (p_2 + q) = p_1 \times p_2 + p_1 \times q$.

(86) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $(p_1 + p_2) \times q = p_1 \times q + p_2 \times q$.

Let us consider p_1, p_2, p_3. The functor $\langle|p_1, p_2, p_3|\rangle$ yields a real number and is defined as follows:

(Def. 6) $\langle|p_1, p_2, p_3|\rangle = |(p_1, p_2 \times p_3)|$.

Next we state several propositions:

(87) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_1, p_1, p_2|\rangle = 0$.

(88) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_2, p_1, p_2|\rangle = 0$.

(89) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_1, p_2, p_2|\rangle = 0$.

(90) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = \langle|p_2, q, p_1|\rangle$.

(91) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = |(p_1 \times p_2, q)|$.

(92) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = |(q \times p_1, p_2)|$.

3. The Differentiation Formulas of Vector Functions in 3-dimensional Euclidean Spaces

Let f_1, f_2, f_3 be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and let t_0 be a real number. The functor $\mathrm{VFuncdiff}(f_1, f_2, f_3, t_0)$ yielding an element of $\mathcal{R}^3$ is defined as follows:

(Def. 7) $\mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) = [{f_1}'(t_0), {f_2}'(t_0), {f_3}'(t_0)]$.

Next we state a number of propositions:

(93) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) = {f_1}'(t_0) \cdot e_1 + {f_2}'(t_0) \cdot e_2 + {f_3}'(t_0) \cdot e_3$.

(94) Suppose that
- (i) f_1 is differentiable in t_0,
- (ii) f_2 is differentiable in t_0,
- (iii) f_3 is differentiable in t_0,
- (iv) g_1 is differentiable in t_0,
- (v) g_2 is differentiable in t_0, and
- (vi) g_3 is differentiable in t_0.

Then $\mathrm{VFuncdiff}(f_1 + g_1, f_2 + g_2, f_3 + g_3, t_0) = \mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) + \mathrm{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(95) Suppose that
- (i) f_1 is differentiable in t_0,
- (ii) f_2 is differentiable in t_0,
- (iii) f_3 is differentiable in t_0,
- (iv) g_1 is differentiable in t_0,
- (v) g_2 is differentiable in t_0, and
- (vi) g_3 is differentiable in t_0.

Then $\mathrm{VFuncdiff}(f_1 - g_1, f_2 - g_2, f_3 - g_3, t_0) = \mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) - \mathrm{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(96) If f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0, then $\mathrm{VFuncdiff}(r\, f_1, r\, f_2, r\, f_3, t_0) = r \cdot \mathrm{VFuncdiff}(f_1, f_2, f_3, t_0)$.

(97) Suppose that
- (i) f_1 is differentiable in t_0,
- (ii) f_2 is differentiable in t_0,
- (iii) f_3 is differentiable in t_0,

(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then $\mathrm{VFuncdiff}(f_1\, g_1, f_2\, g_2, f_3\, g_3, t_0) = [g_1(t_0) \cdot {f_1}'(t_0), g_2(t_0) \cdot {f_2}'(t_0), g_3(t_0) \cdot {f_3}'(t_0)] + [f_1(t_0) \cdot {g_1}'(t_0), f_2(t_0) \cdot {g_2}'(t_0), f_3(t_0) \cdot {g_3}'(t_0)]$.

(98) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in $f_1(t_0)$,
(v) g_2 is differentiable in $f_2(t_0)$, and
(vi) g_3 is differentiable in $f_3(t_0)$.
Then $\mathrm{VFuncdiff}(g_1 \cdot f_1, g_2 \cdot f_2, g_3 \cdot f_3, t_0) = [{g_1}'(f_1(t_0)) \cdot {f_1}'(t_0), {g_2}'(f_2(t_0)) \cdot {f_2}'(t_0), {g_3}'(f_3(t_0)) \cdot {f_3}'(t_0)]$.

(99) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and $g_1(t_0) \neq 0$ and $g_2(t_0) \neq 0$ and $g_3(t_0) \neq 0$. Then $\mathrm{VFuncdiff}(\frac{f_1}{g_1}, \frac{f_2}{g_2}, \frac{f_3}{g_3}, t_0) = [\frac{{f_1}'(t_0) \cdot g_1(t_0) - {g_1}'(t_0) \cdot f_1(t_0)}{g_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0) \cdot g_2(t_0) - {g_2}'(t_0) \cdot f_2(t_0)}{g_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0) \cdot g_3(t_0) - {g_3}'(t_0) \cdot f_3(t_0)}{g_3(t_0)^{\mathbf{2}}}]$.

(100) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and $f_1(t_0) \neq 0$ and $f_2(t_0) \neq 0$ and $f_3(t_0) \neq 0$. Then $\mathrm{VFuncdiff}(\frac{1}{f_1}, \frac{1}{f_2}, \frac{1}{f_3}, t_0) = -[\frac{{f_1}'(t_0)}{f_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)}{f_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)}{f_3(t_0)^{\mathbf{2}}}]$.

(101) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(r\, f_1, r\, f_2, r\, f_3, t_0) = r \cdot {f_1}'(t_0) \cdot e_1 + r \cdot {f_2}'(t_0) \cdot e_2 + r \cdot {f_3}'(t_0) \cdot e_3$.

(102) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then $\mathrm{VFuncdiff}(r\,(f_1 + g_1), r\,(f_2 + g_2), r\,(f_3 + g_3), t_0) =$
$r \cdot \mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) + r \cdot \mathrm{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(103) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and

(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(r\,(f_1 - g_1), r\,(f_2 - g_2), r\,(f_3 - g_3), t_0) =$
$r \cdot$ VFuncdiff$(f_1, f_2, f_3, t_0) - r \cdot$ VFuncdiff(g_1, g_2, g_3, t_0).

(104) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(r\,f_1\,g_1, r\,f_2\,g_2, r\,f_3\,g_3, t_0) = r \cdot [g_1(t_0) \cdot {f_1}'(t_0), g_2(t_0) \cdot {f_2}'(t_0), g_3(t_0) \cdot {f_3}'(t_0)] + r \cdot [f_1(t_0) \cdot {g_1}'(t_0), f_2(t_0) \cdot {g_2}'(t_0), f_3(t_0) \cdot {g_3}'(t_0)]$.

(105) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in $f_1(t_0)$,
(v) g_2 is differentiable in $f_2(t_0)$, and
(vi) g_3 is differentiable in $f_3(t_0)$.
Then VFuncdiff$((r\,g_1) \cdot f_1, (r\,g_2) \cdot f_2, (r\,g_3) \cdot f_3, t_0) = r \cdot [{g_1}'(f_1(t_0)) \cdot {f_1}'(t_0), {g_2}'(f_2(t_0)) \cdot {f_2}'(t_0), {g_3}'(f_3(t_0)) \cdot {f_3}'(t_0)]$.

(106) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and $g_1(t_0) \neq 0$ and $g_2(t_0) \neq 0$ and $g_3(t_0) \neq 0$. Then VFuncdiff$(\frac{r\,f_1}{g_1}, \frac{r\,f_2}{g_2}, \frac{r\,f_3}{g_3}, t_0) = r \cdot [\frac{{f_1}'(t_0) \cdot g_1(t_0) - {g_1}'(t_0) \cdot f_1(t_0)}{g_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0) \cdot g_2(t_0) - {g_2}'(t_0) \cdot f_2(t_0)}{g_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0) \cdot g_3(t_0) - {g_3}'(t_0) \cdot f_3(t_0)}{g_3(t_0)^{\mathbf{2}}}]$.

(107) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and $f_1(t_0) \neq 0$ and $f_2(t_0) \neq 0$ and $f_3(t_0) \neq 0$ and $r \neq 0$. Then VFuncdiff$(\frac{1}{r\,f_1}, \frac{1}{r\,f_2}, \frac{1}{r\,f_3}, t_0) = -\frac{1}{r} \cdot [\frac{{f_1}'(t_0)}{f_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)}{f_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)}{f_3(t_0)^{\mathbf{2}}}]$.

(108) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(f_2\,g_3 - f_3\,g_2, f_3\,g_1 - f_1\,g_3, f_1\,g_2 - f_2\,g_1, t_0) = [f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0), f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0), f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0)] + [{f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0), {f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0), {f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)]$.

(109) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_1\,(f_2\,g_3 - f_3\,g_2), h_2\,(f_3\,g_1 - f_1\,g_3), h_3\,(f_1\,g_2 - f_2\,g_1), t_0) = [{h_1}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0)), {h_2}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)), {h_3}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0))] + [h_1(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0)), h_2(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)), h_3(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0))] + [h_1(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0)), h_2(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)), h_3(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0))]$.

(110) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_2\,f_2\,g_3 - h_3\,f_3\,g_2, h_3\,f_3\,g_1 - h_1\,f_1\,g_3, h_1\,f_1\,g_2 - h_2\,f_2\,g_1, t_0) = [h_2(t_0) \cdot f_2(t_0) \cdot {g_3}'(t_0) - h_3(t_0) \cdot f_3(t_0) \cdot {g_2}'(t_0), h_3(t_0) \cdot f_3(t_0) \cdot {g_1}'(t_0) - h_1(t_0) \cdot f_1(t_0) \cdot {g_3}'(t_0), h_1(t_0) \cdot f_1(t_0) \cdot {g_2}'(t_0) - h_2(t_0) \cdot f_2(t_0) \cdot {g_1}'(t_0)] + [h_2(t_0) \cdot {f_2}'(t_0) \cdot g_3(t_0) - h_3(t_0) \cdot {f_3}'(t_0) \cdot g_2(t_0), h_3(t_0) \cdot {f_3}'(t_0) \cdot g_1(t_0) - h_1(t_0) \cdot {f_1}'(t_0) \cdot g_3(t_0), h_1(t_0) \cdot {f_1}'(t_0) \cdot g_2(t_0) - h_2(t_0) \cdot {f_2}'(t_0) \cdot g_1(t_0)] + [{h_2}'(t_0) \cdot f_2(t_0) \cdot g_3(t_0) - {h_3}'(t_0) \cdot f_3(t_0) \cdot g_2(t_0), {h_3}'(t_0) \cdot f_3(t_0) \cdot g_1(t_0) - {h_1}'(t_0) \cdot f_1(t_0) \cdot g_3(t_0), {h_1}'(t_0) \cdot f_1(t_0) \cdot g_2(t_0) - {h_2}'(t_0) \cdot f_2(t_0) \cdot g_1(t_0)]$.

(111) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_2\,(f_1\,g_2 - f_2\,g_1) - h_3\,(f_3\,g_1 - f_1\,g_3), h_3\,(f_2\,g_3 - f_3\,g_2) - h_1\,(f_1\,g_2 - f_2\,g_1), h_1\,(f_3\,g_1 - f_1\,g_3) - h_2\,(f_2\,g_3 - f_3\,g_2), t_0) = [h_2(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0)) - h_3(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)), h_3(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0)) - h_1(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0)), h_1(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)) - h_2(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0))] + [h_2(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)) - h_3(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)), h_3(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0)) - h_1(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)), h_1(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)) - h_2(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0))] + [{h_2}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0)) - {h_3}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)), {h_3}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0)) - {h_1}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0)), {h_1}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)) - {h_2}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0))]$.

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