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Free Magmas

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Summary. This article introduces the free magma $M(X)$ constructed on a set X [6]. Then, we formalize some theorems about $M(X)$: if f is a function from the set X to a magma N, the free magma $M(X)$ has a unique extension of f to a morphism of $M(X)$ into N and every magma is isomorphic to a magma generated by a set X under a set of relators on $M(X)$. In doing it, the article defines the stable subset under the law of composition of a magma, the submagma, the equivalence relation compatible with the law of composition and the equivalence kernel of a function. We also introduce some schemes on the recursive function.

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The terminology and notation used here have been introduced in the following articles: [19], [12], [7], [2], [14], [4], [8], [9], [17], [15], [1], [3], [10], [5], [20], [21], [13], [18], [16], and [11].

1. Preliminaries

Let X be a set, let f be a function from $\mathbb{N}$ into X, and let n be a natural number. Observe that $f \restriction n$ is transfinite sequence-like.

Let X, x be sets. The 0-sequence $_X(x)$ yielding a finite 0-sequence of X is defined as follows:

(Def. 1) The 0-sequence $_X(x) = \begin{cases} x, & \text{if } x \text{ is a finite 0-sequence of } X, \\ \langle\rangle_X, & \text{otherwise.} \end{cases}$

Let X be a non empty set, let f be a function from X^ω into X, and let c be a finite 0-sequence of X. Then $f(c)$ is an element of X.

One can prove the following proposition

(1) For all sets X, Y, Z such that $Y \subseteq$ the universe of X and $Z \subseteq$ the universe of X holds $Y \times Z \subseteq$ the universe of X.

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In this article we present several logical schemes. The scheme *FuncRecursiveUniq* deals with a unary functor $\mathcal{F}$ yielding a set and functions $\mathcal{A}$, $\mathcal{B}$, and states that:

$\mathcal{A} = \mathcal{B}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$ and for every natural number n holds $\mathcal{A}(n) = \mathcal{F}(\mathcal{A}\restriction n)$, and
- $\operatorname{dom} \mathcal{B} = \mathbb{N}$ and for every natural number n holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\restriction n)$.

The scheme *FuncRecursiveExist* deals with a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and for every natural number n holds $f(n) = \mathcal{F}(f\restriction n)$

for all values of the parameter.

The scheme *FuncRecursiveUniqu2* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and functions $\mathcal{B}$, $\mathcal{C}$ from $\mathbb{N}$ into $\mathcal{A}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters meet the following requirements:

- For every element n of $\mathbb{N}$ holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\restriction n)$, and
- For every element n of $\mathbb{N}$ holds $\mathcal{C}(n) = \mathcal{F}(\mathcal{C}\restriction n)$.

The scheme *FuncRecursiveExist2* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that for every natural number n holds $f(n) = \mathcal{F}(f\restriction n)$

for all values of the parameters.

Let f, g be functions. We say that f extends g if and only if:

(Def. 2) $\operatorname{dom} g \subseteq \operatorname{dom} f$ and $f \approx g$.

Let us note that there exists a multiplicative magma which is empty.

2. Equivalence Relations and Relators

Let M be a multiplicative magma and let R be an equivalence relation of M. We say that R is compatible if and only if:

(Def. 3) For all elements v, v', w, w' of M such that $v \in [v']_R$ and $w \in [w']_R$ holds $v \cdot w \in [v' \cdot w']_R$.

Let M be a multiplicative magma. Observe that $\nabla_{\text{the carrier of } M}$ is compatible.

Let M be a multiplicative magma. Observe that there exists an equivalence relation of M which is compatible.

One can prove the following proposition

(2) Let M be a multiplicative magma and R be an equivalence relation of M. Then R is compatible if and only if for all elements v, v', w, w' of M such that $[v]_R = [v']_R$ and $[w]_R = [w']_R$ holds $[v \cdot w]_R = [v' \cdot w']_R$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. The functor $\circ_R$ yielding a binary operation on Classes R is defined as follows:

(Def. 4)(i) For all elements x, y of Classes R and for all elements v, w of M such that $x = [v]_R$ and $y = [w]_R$ holds $(\circ_R)(x, y) = [v \cdot w]_R$ if M is non empty,

(ii) $\circ_R = \emptyset$, otherwise.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. The functor ${}^{M}/_{R}$ yielding a multiplicative magma is defined as follows:

(Def. 5) ${}^{M}/_{R} = \langle \text{Classes}\, R, \circ_R \rangle$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. Observe that ${}^{M}/_{R}$ is strict.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. One can check that ${}^{M}/_{R}$ is non empty.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. The canonical homomorphism onto cosets of R yields a function from M into ${}^{M}/_{R}$ and is defined by:

(Def. 6) For every element v of M holds (the canonical homomorphism onto cosets of R)$(v) = [v]_R$.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. Note that the canonical homomorphism onto cosets of R is multiplicative.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. Note that the canonical homomorphism onto cosets of R is onto.

Let M be a multiplicative magma. A function is called a relators of M if:

(Def. 7) $\operatorname{rng} \mathrm{it} \subseteq$ (the carrier of M) $\times$ (the carrier of M).

Let M be a multiplicative magma and let r be a relators of M. The equivalence relation of r yielding an equivalence relation of M is defined by the condition (Def. 8).

(Def. 8) The equivalence relation of $r = \bigcap\{R; R$ ranges over compatible equivalence relations of M: $\bigwedge_{i\,:\,\mathrm{set}} \bigwedge_{v,w\,:\,\text{element of } M} (i \in \operatorname{dom} r \;\wedge\; r(i) = \langle v, w \rangle \;\Rightarrow\; v \in [w]_R)\}$.

Next we state the proposition

(3) Let M be a multiplicative magma, r be a relators of M, and R be a compatible equivalence relation of M. Suppose that for every set i and

for all elements v, w of M such that $i \in \operatorname{dom} r$ and $r(i) = \langle v, w \rangle$ holds $v \in [w]_R$. Then the equivalence relation of $r \subseteq R$.

Let M be a multiplicative magma and let r be a relators of M. Note that the equivalence relation of r is compatible.

Let X, Y be sets and let f be a function from X into Y. The equivalence kernel of f yielding an equivalence relation of X is defined as follows:

(Def. 9) For all sets x, y holds $\langle x, y \rangle \in$ the equivalence kernel of f iff $x, y \in X$ and $f(x) = f(y)$.

In the sequel M, N are non empty multiplicative magmas and f is a function from M into N.

The following propositions are true:

(4) If f is multiplicative, then the equivalence kernel of f is compatible.

(5) Suppose f is multiplicative. Then there exists a relators r of M such that the equivalence kernel of $f =$ the equivalence relation of r.

3. Submagmas and Stable Subsets

Let M be a multiplicative magma. A multiplicative magma is said to be a submagma of M if it satisfies the conditions (Def. 10).

(Def. 10)(i) The carrier of it $\subseteq$ the carrier of M, and

(ii) the multiplication of it $=$ (the multiplication of M) $\restriction$ (the carrier of it).

Let M be a multiplicative magma. One can check that there exists a submagma of M which is strict.

Let M be a non empty multiplicative magma. Note that there exists a submagma of M which is non empty.

In the sequel M denotes a multiplicative magma and N, K denote submagmas of M.

One can prove the following propositions:

(6) Suppose N is a submagma of K and K is a submagma of N. Then the multiplicative magma of $N =$ the multiplicative magma of K.

(7) Suppose the carrier of $N =$ the carrier of M. Then the multiplicative magma of $N =$ the multiplicative magma of M.

Let M be a multiplicative magma and let A be a subset of M. We say that A is stable if and only if:

(Def. 11) For all elements v, w of M such that $v, w \in A$ holds $v \cdot w \in A$.

Let M be a multiplicative magma. One can check that there exists a subset of M which is stable.

We now state the proposition

(8) The carrier of N is a stable subset of M.

Let M be a multiplicative magma and let N be a submagma of M. Note that the carrier of N is stable.

We now state the proposition

(9) Let F be a function. Suppose that for every set i such that $i \in \operatorname{dom} F$ holds $F(i)$ is a stable subset of M. Then $\bigcap F$ is a stable subset of M.

For simplicity, we adopt the following convention: M, N are non empty multiplicative magmas, A is a subset of M, f, g are functions from M into N, X is a stable subset of M, and Y is a stable subset of N.

Next we state four propositions:

(10) A is stable iff $A \cdot A \subseteq A$.

(11) If f is multiplicative, then $f^\circ X$ is a stable subset of N.

(12) If f is multiplicative, then $f^{-1}(Y)$ is a stable subset of M.

(13) If f is multiplicative and g is multiplicative, then $\{v \in M\colon f(v) = g(v)\}$ is a stable subset of M.

Let M be a multiplicative magma and let A be a stable subset of M. The multiplication induced by A yields a binary operation on A and is defined by:

(Def. 12) The multiplication induced by $A =$ (the multiplication of M) $\restriction A$.

Let M be a multiplicative magma and let A be a subset of M. The submagma generated by A yielding a strict submagma of M is defined by the conditions (Def. 13).

(Def. 13)(i) $A \subseteq$ the carrier of the submagma generated by A, and

(ii) for every strict submagma N of M such that $A \subseteq$ the carrier of N holds the submagma generated by A is a submagma of N.

We now state the proposition

(14) Let M be a multiplicative magma and A be a subset of M. Then A is empty if and only if the submagma generated by A is empty.

Let M be a multiplicative magma and let A be an empty subset of M. Note that the submagma generated by A is empty.

The following proposition is true

(15) Let M, N be non empty multiplicative magmas, f be a function from M into N, and X be a subset of M. Suppose f is multiplicative. Then f°(the carrier of the submagma generated by X) = the carrier of the submagma generated by $f^\circ X$.

4. Free Magmas

Let X be a set. The free magma sequence of X yielding a function from $\mathbb{N}$ into $2^{\text{the universe of } X \cup \mathbb{N}}$ is defined by the conditions (Def. 14).

(Def. 14)(i) (The free magma sequence of X)$(0) = \emptyset$,

(ii) (the free magma sequence of X)$(1) = X$, and

(iii) for every natural number n such that $n \geq 2$ there exists a finite sequence f_1 such that $\operatorname{len} f_1 = n - 1$ and for every natural number p such that $p \geq 1$ and $p \leq n - 1$ holds $f_1(p) =$ (the free magma sequence of X)$(p) \times$ (the free magma sequence of X)$(n - p)$ and (the free magma sequence of X)$(n) = \bigcup \text{disjoint}\, f_1$.

Let X be a set and let n be a natural number. The functor $\mathrm{M_n}(X)$ is defined by:

(Def. 15) $\mathrm{M_n}(X) =$ (the free magma sequence of X)(n).

Let X be a non empty set and let n be a non zero natural number. Observe that $\mathrm{M_n}(X)$ is non empty.

In the sequel X is a set.

We now state four propositions:

(16) $\mathrm{M_0}(X) = \emptyset$.

(17) $\mathrm{M_1}(X) = X$.

(18) $\mathrm{M_2}(X) = X \times X \times \{1\}$.

(19) $\mathrm{M_3}(X) = X \times (X \times X \times \{1\}) \times \{1\} \cup X \times X \times \{1\} \times X \times \{2\}$.

We adopt the following convention: x, y, Y are sets and n, m, p are elements of $\mathbb{N}$.

One can prove the following propositions:

(20) Suppose $n \geq 2$. Then there exists a finite sequence f_1 such that $\operatorname{len} f_1 = n-1$ and for every p such that $p \geq 1$ and $p \leq n-1$ holds $f_1(p) = \mathrm{M_p}(X) \times \mathrm{M_{n-'p}}(X)$ and $\mathrm{M_n}(X) = \bigcup \text{disjoint}\, f_1$.

(21) Suppose $n \geq 2$ and $x \in \mathrm{M_n}(X)$. Then there exist p, m such that $x_{\mathbf{2}} = p$ and $1 \leq p \leq n-1$ and $(x_{\mathbf{1}})_{\mathbf{1}} \in \mathrm{M_p}(X)$ and $(x_{\mathbf{1}})_{\mathbf{2}} \in \mathrm{M_m}(X)$ and $n = p+m$ and $x \in \mathrm{M_p}(X) \times \mathrm{M_m}(X) \times \{p\}$.

(22) If $x \in \mathrm{M_n}(X)$ and $y \in \mathrm{M_m}(X)$, then $\langle \langle x, y \rangle, n \rangle \in \mathrm{M_{n+m}}(X)$.

(23) If $X \subseteq Y$, then $\mathrm{M_n}(X) \subseteq \mathrm{M_n}(Y)$.

Let X be a set. The carrier of free magma on X is defined as follows:

(Def. 16) The carrier of free magma on $X = \bigcup \text{disjoint}(($the free magma sequence of $X) \restriction \mathbb{N}^+)$.

One can prove the following proposition

(24) $X = \emptyset$ iff the carrier of free magma on $X = \emptyset$.

Let X be an empty set. Observe that the carrier of free magma on X is empty.

Let X be a non empty set. One can verify that the carrier of free magma on X is non empty. Let w be an element of the carrier of free magma on X. Observe that $w_{\mathbf{2}}$ is non zero and natural.

We now state four propositions:

(25) For every non empty set X and for every element w of the carrier of free magma on X holds $w \in \mathrm{M}_{w_{\mathbf{2}}}(X) \times \{w_{\mathbf{2}}\}$.

(26) Let X be a non empty set and v, w be elements of the carrier of free magma on X. Then $\langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, v_{\mathbf{2}} + w_{\mathbf{2}}\rangle$ is an element of the carrier of free magma on X.

(27) If $X \subseteq Y$, then the carrier of free magma on $X \subseteq$ the carrier of free magma on Y.

(28) If $n > 0$, then $\mathrm{M}_{\mathrm{n}}(X) \times \{n\} \subseteq$ the carrier of free magma on X.

Let X be a set. The multiplication of free magma on X yields a binary operation on the carrier of free magma on X and is defined as follows:

(Def. 17)(i) For all elements v, w of the carrier of free magma on X and for all n, m such that $n = v_{\mathbf{2}}$ and $m = w_{\mathbf{2}}$ holds (the multiplication of free magma on X)$(v, w) = \langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, n + m\rangle$ if X is non empty,

(ii) the multiplication of free magma on $X = \emptyset$, otherwise.

Let X be a set. The functor $\mathrm{M}(X)$ yields a multiplicative magma and is defined by:

(Def. 18) $\mathrm{M}(X) = \langle$the carrier of free magma on X, the multiplication of free magma on $X\rangle$.

Let X be a set. Note that $\mathrm{M}(X)$ is strict.

Let X be an empty set. One can verify that $\mathrm{M}(X)$ is empty.

Let X be a non empty set. Note that $\mathrm{M}(X)$ is non empty. Let w be an element of $\mathrm{M}(X)$. One can verify that $w_{\mathbf{2}}$ is non zero and natural.

The following proposition is true

(29) For every set X and for every subset S of X holds $\mathrm{M}(S)$ is a submagma of $\mathrm{M}(X)$.

Let X be a set and let w be an element of $\mathrm{M}(X)$. The functor length w yields a natural number and is defined by:

(Def. 19) $\operatorname{length} w = \begin{cases} w_{\mathbf{2}}, & \text{if } X \text{ is non empty,} \\ 0, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(30) $X = \{w_{\mathbf{1}}; w$ ranges over elements of $\mathrm{M}(X)$: $\operatorname{length} w = 1\}$.

In the sequel v, v_1, v_2, w, w_1, w_2 denote elements of $\mathrm{M}(X)$.

One can prove the following propositions:

(31) If X is non empty, then $v \cdot w = \langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, \operatorname{length} v + \operatorname{length} w\rangle$.

(32) If X is non empty, then $v = \langle v_{\mathbf{1}}, v_{\mathbf{2}}\rangle$ and $\operatorname{length} v \geq 1$.

(33) $\operatorname{length}(v \cdot w) = \operatorname{length} v + \operatorname{length} w$.

(34) If $\operatorname{length} w \geq 2$, then there exist w_1, w_2 such that $w = w_1 \cdot w_2$ and $\operatorname{length} w_1 < \operatorname{length} w$ and $\operatorname{length} w_2 < \operatorname{length} w$.

(35) If $v_1 \cdot v_2 = w_1 \cdot w_2$, then $v_1 = w_1$ and $v_2 = w_2$.

Let X be a set and let n be a natural number. The n-canonical image of X yields a function from $\mathrm{M_n}(X)$ into $\mathrm{M}(X)$ and is defined as follows:

(Def. 20)(i) For every x such that $x \in \operatorname{dom}$ (the n-canonical image of X) holds (the n-canonical image of X)$(x) = \langle x, n\rangle$ if $n > 0$,

(ii) the n-canonical image of $X = \emptyset$, otherwise.

Let X be a set and let n be a natural number. Observe that the n-canonical image of X is one-to-one.

Let X be a non empty set. Observe that the 1-canonical image of X

In the sequel X, Y, Z are non empty sets.

Next we state three propositions:

(36) For every subset A of $\mathrm{M}(X)$ such that $A =$ (the 1-canonical image of $X)^{\circ}X$ holds $\mathrm{M}(X) =$ the submagma generated by A.

(37) Let R be a compatible equivalence relation of $\mathrm{M}(X)$. Then ${}^{\mathrm{M}(X)}/_{R} =$ the submagma generated by (the canonical homomorphism onto cosets of $R)^{\circ}$(the 1-canonical image of $X)^{\circ}X$.

(38) For every function f from X into Y holds (the 1-canonical image of $Y) \cdot f$ is a function from X into $\mathrm{M}(Y)$.

Let X be a non empty set, let M be a non empty multiplicative magma, let n, m be non zero natural numbers, let f be a function from $\mathrm{M_n}(X)$ into M, and let g be a function from $\mathrm{M_m}(X)$ into M. The functor $f \times g$ yielding a function from $\mathrm{M_n}(X) \times \mathrm{M_m}(X) \times \{n\}$ into M is defined by the condition (Def. 21).

(Def. 21) Let x be an element of $\mathrm{M_n}(X) \times \mathrm{M_m}(X) \times \{n\}$, y be an element of $\mathrm{M_n}(X)$, and z be an element of $\mathrm{M_m}(X)$. If $y = (x_{\mathbf{1}})_{\mathbf{1}}$ and $z = (x_{\mathbf{1}})_{\mathbf{2}}$, then $(f \times g)(x) = f(y) \cdot g(z)$.

In the sequel M is a non empty multiplicative magma.

One can prove the following propositions:

(39) Let f be a function from X into M. Then there exists a function h from $\mathrm{M}(X)$ into M such that h is multiplicative and h extends $f \cdot$ (the 1-canonical image of $X)^{-1}$.

(40) Let f be a function from X into M and h, g be functions from $\mathrm{M}(X)$ into M. Suppose that

(i) h is multiplicative,

(ii) h extends $f \cdot$ (the 1-canonical image of $X)^{-1}$,

(iii) g is multiplicative, and

(iv) g extends $f \cdot (\text{the 1-canonical image of } X)^{-1}$.
Then $h = g$.

For simplicity, we adopt the following rules: M, N are non empty multiplicative magmas, f is a function from M into N, H is a non empty submagma of N, and R is a compatible equivalence relation of M.

We now state three propositions:

(41) Suppose f is multiplicative and the carrier of $H = \operatorname{rng} f$ and $R =$ the equivalence kernel of f. Then there exists a function g from ${}^{M}/_{R}$ into H such that $f = g \cdot$ the canonical homomorphism onto cosets of R and g is bijective and multiplicative.

(42) Let g_1, g_2 be functions from ${}^{M}/_{R}$ into N. Suppose $g_1 \cdot$ the canonical homomorphism onto cosets of $R = g_2 \cdot$ the canonical homomorphism onto cosets of R. Then $g_1 = g_2$.

(43) Let M be a non empty multiplicative magma. Then there exists a non empty set X and there exists a relators r of $\mathrm{M}(X)$ such that there exists a function from ${}^{\mathrm{M}(X)}/_{\text{the equivalence relation of } r}$ into M which is bijective and multiplicative.

Let X, Y be non empty sets and let f be a function from X into Y. The functor $\mathbf{M}(f)$ yields a function from $\mathrm{M}(X)$ into $\mathrm{M}(Y)$ and is defined by:

(Def. 22) $\mathbf{M}(f)$ is multiplicative and $\mathbf{M}(f)$ extends $(\text{the 1-canonical image of } Y) \cdot f \cdot (\text{the 1-canonical image of } X)^{-1}$.

Let X, Y be non empty sets and let f be a function from X into Y. One can verify that $\mathbf{M}(f)$ is multiplicative.

In the sequel f denotes a function from X into Y and g denotes a function from Y into Z.

Next we state several propositions:

(44) $\mathbf{M}(g \cdot f) = \mathbf{M}(g) \cdot \mathbf{M}(f)$.

(45) For every function g from X into Z such that $Y \subseteq Z$ and $f = g$ holds $\mathbf{M}(f) = \mathbf{M}(g)$.

(46) $\mathbf{M}(\mathrm{id}_X) = \mathrm{id}_{\operatorname{dom} \mathbf{M}(f)}$.

(47) If f is one-to-one, then $\mathbf{M}(f)$ is one-to-one.

(48) If f is onto, then $\mathbf{M}(f)$ is onto.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[6] Nicolas Bourbaki. *Elements of Mathematics. Algebra I. Chapters 1-3*. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[12] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Formalized Mathematics*, 4(**1**):109–113, 1993.
[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[14] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[16] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[17] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[19] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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