

Free Magmas

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Summary. This article introduces the free magma $M(X)$ constructed on a set X [6]. Then, we formalize some theorems about $M(X)$: if f is a function from the set X to a magma N , the free magma $M(X)$ has a unique extension of f to a morphism of $M(X)$ into N and every magma is isomorphic to a magma generated by a set X under a set of relators on $M(X)$. In doing it, the article defines the stable subset under the law of composition of a magma, the submagma, the equivalence relation compatible with the law of composition and the equivalence kernel of a function. We also introduce some schemes on the recursive function.

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The terminology and notation used here have been introduced in the following articles: [19], [12], [7], [2], [14], [4], [8], [9], [17], [15], [1], [3], [10], [5], [20], [21], [13], [18], [16], and [11].

1. PRELIMINARIES

Let X be a set, let f be a function from \mathbb{N} into X , and let n be a natural number. Observe that $f \upharpoonright n$ is transfinite sequence-like.

Let X, x be sets. The 0-sequence $x(x)$ yielding a finite 0-sequence of X is defined as follows:

(Def. 1) The 0-sequence $x(x) = \begin{cases} x, & \text{if } x \text{ is a finite 0-sequence of } X, \\ \langle \rangle_X, & \text{otherwise.} \end{cases}$

Let X be a non empty set, let f be a function from X^ω into X , and let c be a finite 0-sequence of X . Then $f(c)$ is an element of X .

One can prove the following proposition

- (1) For all sets X, Y, Z such that $Y \subseteq$ the universe of X and $Z \subseteq$ the universe of X holds $Y \times Z \subseteq$ the universe of X .

In this article we present several logical schemes. The scheme *FuncRecursiveUniq* deals with a unary functor \mathcal{F} yielding a set and functions \mathcal{A} , \mathcal{B} , and states that:

$$\mathcal{A} = \mathcal{B}$$

provided the parameters satisfy the following conditions:

- $\text{dom } \mathcal{A} = \mathbb{N}$ and for every natural number n holds $\mathcal{A}(n) = \mathcal{F}(\mathcal{A}\upharpoonright n)$,
and
- $\text{dom } \mathcal{B} = \mathbb{N}$ and for every natural number n holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\upharpoonright n)$.

The scheme *FuncRecursiveExist* deals with a unary functor \mathcal{F} yielding a set, and states that:

There exists a function f such that $\text{dom } f = \mathbb{N}$ and for every natural number n holds $f(n) = \mathcal{F}(f\upharpoonright n)$

for all values of the parameter.

The scheme *FuncRecursiveUniqu2* deals with a non empty set \mathcal{A} , a unary functor \mathcal{F} yielding an element of \mathcal{A} , and functions \mathcal{B} , \mathcal{C} from \mathbb{N} into \mathcal{A} , and states that:

$$\mathcal{B} = \mathcal{C}$$

provided the parameters meet the following requirements:

- For every element n of \mathbb{N} holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B}\upharpoonright n)$, and
- For every element n of \mathbb{N} holds $\mathcal{C}(n) = \mathcal{F}(\mathcal{C}\upharpoonright n)$.

The scheme *FuncRecursiveExist2* deals with a non empty set \mathcal{A} and a unary functor \mathcal{F} yielding an element of \mathcal{A} , and states that:

There exists a function f from \mathbb{N} into \mathcal{A} such that for every natural number n holds $f(n) = \mathcal{F}(f\upharpoonright n)$

for all values of the parameters.

Let f, g be functions. We say that f extends g if and only if:

(Def. 2) $\text{dom } g \subseteq \text{dom } f$ and $f \approx g$.

Let us note that there exists a multiplicative magma which is empty.

2. EQUIVALENCE RELATIONS AND RELATORS

Let M be a multiplicative magma and let R be an equivalence relation of M . We say that R is compatible if and only if:

(Def. 3) For all elements v, v', w, w' of M such that $v \in [v']_R$ and $w \in [w']_R$ holds $v \cdot w \in [v' \cdot w']_R$.

Let M be a multiplicative magma. Observe that $\nabla_{\text{the carrier of } M}$ is compatible.

Let M be a multiplicative magma. Observe that there exists an equivalence relation of M which is compatible.

One can prove the following proposition

- (2) Let M be a multiplicative magma and R be an equivalence relation of M . Then R is compatible if and only if for all elements v, v', w, w' of M such that $[v]_R = [v']_R$ and $[w]_R = [w']_R$ holds $[v \cdot w]_R = [v' \cdot w']_R$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M . The functor \circ_R yielding a binary operation on Classes R is defined as follows:

- (Def. 4)(i) For all elements x, y of Classes R and for all elements v, w of M such that $x = [v]_R$ and $y = [w]_R$ holds $(\circ_R)(x, y) = [v \cdot w]_R$ if M is non empty,
 (ii) $\circ_R = \emptyset$, otherwise.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M . The functor $^M/R$ yielding a multiplicative magma is defined as follows:

- (Def. 5) $^M/R = \langle \text{Classes } R, \circ_R \rangle$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M . Observe that $^M/R$ is strict.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M . One can check that $^M/R$ is non empty.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M . The canonical homomorphism onto cosets of R yields a function from M into $^M/R$ and is defined by:

- (Def. 6) For every element v of M holds (the canonical homomorphism onto cosets of R)(v) = $[v]_R$.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M . Note that the canonical homomorphism onto cosets of R is multiplicative.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M . Note that the canonical homomorphism onto cosets of R is onto.

Let M be a multiplicative magma. A function is called a relators of M if:

- (Def. 7) $\text{rng } r \subseteq (\text{the carrier of } M) \times (\text{the carrier of } M)$.

Let M be a multiplicative magma and let r be a relators of M . The equivalence relation of r yielding an equivalence relation of M is defined by the condition (Def. 8).

- (Def. 8) The equivalence relation of $r = \bigcap \{R; R \text{ ranges over compatible equivalence relations of } M: \bigwedge_{i:\text{set}} \bigwedge_{v,w:\text{element of } M} (i \in \text{dom } r \wedge r(i) = \langle v, w \rangle \Rightarrow v \in [w]_R)\}$.

Next we state the proposition

- (3) Let M be a multiplicative magma, r be a relators of M , and R be a compatible equivalence relation of M . Suppose that for every set i and

for all elements v, w of M such that $i \in \text{dom } r$ and $r(i) = \langle v, w \rangle$ holds $v \in [w]_R$. Then the equivalence relation of $r \subseteq R$.

Let M be a multiplicative magma and let r be a relators of M . Note that the equivalence relation of r is compatible.

Let X, Y be sets and let f be a function from X into Y . The equivalence kernel of f yielding an equivalence relation of X is defined as follows:

(Def. 9) For all sets x, y holds $\langle x, y \rangle \in$ the equivalence kernel of f iff $x, y \in X$ and $f(x) = f(y)$.

In the sequel M, N are non empty multiplicative magmas and f is a function from M into N .

The following propositions are true:

- (4) If f is multiplicative, then the equivalence kernel of f is compatible.
- (5) Suppose f is multiplicative. Then there exists a relators r of M such that the equivalence kernel of $f =$ the equivalence relation of r .

3. SUBMAGMAS AND STABLE SUBSETS

Let M be a multiplicative magma. A multiplicative magma is said to be a submagma of M if it satisfies the conditions (Def. 10).

(Def. 10)(i) The carrier of it \subseteq the carrier of M , and
(ii) the multiplication of it = (the multiplication of M) \upharpoonright (the carrier of it).

Let M be a multiplicative magma. One can check that there exists a submagma of M which is strict.

Let M be a non empty multiplicative magma. Note that there exists a submagma of M which is non empty.

In the sequel M denotes a multiplicative magma and N, K denote submagmas of M .

One can prove the following propositions:

- (6) Suppose N is a submagma of K and K is a submagma of N . Then the multiplicative magma of $N =$ the multiplicative magma of K .
- (7) Suppose the carrier of $N =$ the carrier of M . Then the multiplicative magma of $N =$ the multiplicative magma of M .

Let M be a multiplicative magma and let A be a subset of M . We say that A is stable if and only if:

(Def. 11) For all elements v, w of M such that $v, w \in A$ holds $v \cdot w \in A$.

Let M be a multiplicative magma. One can check that there exists a subset of M which is stable.

We now state the proposition

- (8) The carrier of N is a stable subset of M .

Let M be a multiplicative magma and let N be a submagma of M . Note that the carrier of N is stable.

We now state the proposition

- (9) Let F be a function. Suppose that for every set i such that $i \in \text{dom } F$ holds $F(i)$ is a stable subset of M . Then $\bigcap F$ is a stable subset of M .

For simplicity, we adopt the following convention: M, N are non empty multiplicative magmas, A is a subset of M , f, g are functions from M into N , X is a stable subset of M , and Y is a stable subset of N .

Next we state four propositions:

- (10) A is stable iff $A \cdot A \subseteq A$.
 (11) If f is multiplicative, then $f^\circ X$ is a stable subset of N .
 (12) If f is multiplicative, then $f^{-1}(Y)$ is a stable subset of M .
 (13) If f is multiplicative and g is multiplicative, then $\{v \in M: f(v) = g(v)\}$ is a stable subset of M .

Let M be a multiplicative magma and let A be a stable subset of M . The multiplication induced by A yields a binary operation on A and is defined by:

(Def. 12) The multiplication induced by $A = (\text{the multiplication of } M) \upharpoonright A$.

Let M be a multiplicative magma and let A be a subset of M . The submagma generated by A yielding a strict submagma of M is defined by the conditions (Def. 13).

- (Def. 13)(i) $A \subseteq$ the carrier of the submagma generated by A , and
 (ii) for every strict submagma N of M such that $A \subseteq$ the carrier of N holds the submagma generated by A is a submagma of N .

We now state the proposition

- (14) Let M be a multiplicative magma and A be a subset of M . Then A is empty if and only if the submagma generated by A is empty.

Let M be a multiplicative magma and let A be an empty subset of M . Note that the submagma generated by A is empty.

The following proposition is true

- (15) Let M, N be non empty multiplicative magmas, f be a function from M into N , and X be a subset of M . Suppose f is multiplicative. Then $f^\circ(\text{the carrier of the submagma generated by } X) = \text{the carrier of the submagma generated by } f^\circ X$.

4. FREE MAGMAS

Let X be a set. The free magma sequence of X yielding a function from \mathbb{N} into $2^{\text{the universe of } X \cup \mathbb{N}}$ is defined by the conditions (Def. 14).

- (Def. 14)(i) (The free magma sequence of X)(0) = \emptyset ,
(ii) (the free magma sequence of X)(1) = X , and
(iii) for every natural number n such that $n \geq 2$ there exists a finite sequence f_1 such that $\text{len } f_1 = n - 1$ and for every natural number p such that $p \geq 1$ and $p \leq n - 1$ holds $f_1(p) = (\text{the free magma sequence of } X)(p) \times (\text{the free magma sequence of } X)(n - p)$ and $(\text{the free magma sequence of } X)(n) = \bigcup \text{disjoint } f_1$.

Let X be a set and let n be a natural number. The functor $M_n(X)$ is defined by:

- (Def. 15) $M_n(X) = (\text{the free magma sequence of } X)(n)$.

Let X be a non empty set and let n be a non zero natural number. Observe that $M_n(X)$ is non empty.

In the sequel X is a set.

We now state four propositions:

- (16) $M_0(X) = \emptyset$.
(17) $M_1(X) = X$.
(18) $M_2(X) = X \times X \times \{1\}$.
(19) $M_3(X) = X \times (X \times X \times \{1\}) \times \{1\} \cup X \times X \times \{1\} \times X \times \{2\}$.

We adopt the following convention: x, y, Y are sets and n, m, p are elements of \mathbb{N} .

One can prove the following propositions:

- (20) Suppose $n \geq 2$. Then there exists a finite sequence f_1 such that $\text{len } f_1 = n - 1$ and for every p such that $p \geq 1$ and $p \leq n - 1$ holds $f_1(p) = M_p(X) \times M_{n-p}(X)$ and $M_n(X) = \bigcup \text{disjoint } f_1$.
(21) Suppose $n \geq 2$ and $x \in M_n(X)$. Then there exist p, m such that $x_2 = p$ and $1 \leq p \leq n - 1$ and $(x_1)_1 \in M_p(X)$ and $(x_1)_2 \in M_m(X)$ and $n = p + m$ and $x \in M_p(X) \times M_m(X) \times \{p\}$.
(22) If $x \in M_n(X)$ and $y \in M_m(X)$, then $\langle \langle x, y \rangle, n \rangle \in M_{n+m}(X)$.
(23) If $X \subseteq Y$, then $M_n(X) \subseteq M_n(Y)$.

Let X be a set. The carrier of free magma on X is defined as follows:

- (Def. 16) The carrier of free magma on $X = \bigcup \text{disjoint}((\text{the free magma sequence of } X) \upharpoonright \mathbb{N}^+)$.

One can prove the following proposition

- (24) $X = \emptyset$ iff the carrier of free magma on $X = \emptyset$.

Let X be an empty set. Observe that the carrier of free magma on X is empty.

Let X be a non empty set. One can verify that the carrier of free magma on X is non empty. Let w be an element of the carrier of free magma on X . Observe that w_2 is non zero and natural.

We now state four propositions:

- (25) For every non empty set X and for every element w of the carrier of free magma on X holds $w \in M_{w_2}(X) \times \{w_2\}$.
- (26) Let X be a non empty set and v, w be elements of the carrier of free magma on X . Then $\langle \langle \langle v_1, w_1 \rangle, v_2 \rangle, v_2 + w_2 \rangle$ is an element of the carrier of free magma on X .
- (27) If $X \subseteq Y$, then the carrier of free magma on $X \subseteq$ the carrier of free magma on Y .
- (28) If $n > 0$, then $M_n(X) \times \{n\} \subseteq$ the carrier of free magma on X .

Let X be a set. The multiplication of free magma on X yields a binary operation on the carrier of free magma on X and is defined as follows:

- (Def. 17)(i) For all elements v, w of the carrier of free magma on X and for all n, m such that $n = v_2$ and $m = w_2$ holds (the multiplication of free magma on X)(v, w) = $\langle \langle \langle v_1, w_1 \rangle, v_2 \rangle, n + m \rangle$ if X is non empty,
- (ii) the multiplication of free magma on $X = \emptyset$, otherwise.

Let X be a set. The functor $M(X)$ yields a multiplicative magma and is defined by:

- (Def. 18) $M(X) = \langle$ the carrier of free magma on X , the multiplication of free magma on $X \rangle$.

Let X be a set. Note that $M(X)$ is strict.

Let X be an empty set. One can verify that $M(X)$ is empty.

Let X be a non empty set. Note that $M(X)$ is non empty. Let w be an element of $M(X)$. One can verify that w_2 is non zero and natural.

The following proposition is true

- (29) For every set X and for every subset S of X holds $M(S)$ is a submagma of $M(X)$.

Let X be a set and let w be an element of $M(X)$. The functor $\text{length } w$ yields a natural number and is defined by:

- (Def. 19) $\text{length } w = \begin{cases} w_2, & \text{if } X \text{ is non empty,} \\ 0, & \text{otherwise.} \end{cases}$

One can prove the following proposition

- (30) $X = \{w_1; w \text{ ranges over elements of } M(X): \text{length } w = 1\}$.

In the sequel v, v_1, v_2, w, w_1, w_2 denote elements of $M(X)$.

One can prove the following propositions:

- (31) If X is non empty, then $v \cdot w = \langle \langle v_1, w_1 \rangle, v_2 \rangle$, $\text{length } v + \text{length } w$.
- (32) If X is non empty, then $v = \langle v_1, v_2 \rangle$ and $\text{length } v \geq 1$.
- (33) $\text{length}(v \cdot w) = \text{length } v + \text{length } w$.
- (34) If $\text{length } w \geq 2$, then there exist w_1, w_2 such that $w = w_1 \cdot w_2$ and $\text{length } w_1 < \text{length } w$ and $\text{length } w_2 < \text{length } w$.
- (35) If $v_1 \cdot v_2 = w_1 \cdot w_2$, then $v_1 = w_1$ and $v_2 = w_2$.

Let X be a set and let n be a natural number. The n -canonical image of X yields a function from $M_n(X)$ into $M(X)$ and is defined as follows:

- (Def. 20)(i) For every x such that $x \in \text{dom}$ (the n -canonical image of X) holds
 (the n -canonical image of X)(x) = $\langle x, n \rangle$ if $n > 0$,
- (ii) the n -canonical image of $X = \emptyset$, otherwise.

Let X be a set and let n be a natural number. Observe that the n -canonical image of X is one-to-one.

Let X be a non empty set. Observe that the 1-canonical image of X

In the sequel X, Y, Z are non empty sets.

Next we state three propositions:

- (36) For every subset A of $M(X)$ such that $A = (\text{the 1-canonical image of } X)^\circ X$ holds $M(X) = \text{the submagma generated by } A$.
- (37) Let R be a compatible equivalence relation of $M(X)$. Then $M(X)/R = \text{the submagma generated by } (\text{the canonical homomorphism onto cosets of } R)^\circ (\text{the 1-canonical image of } X)^\circ X$.
- (38) For every function f from X into Y holds $(\text{the 1-canonical image of } Y) \cdot f$ is a function from X into $M(Y)$.

Let X be a non empty set, let M be a non empty multiplicative magma, let n, m be non zero natural numbers, let f be a function from $M_n(X)$ into M , and let g be a function from $M_m(X)$ into M . The functor $f \times g$ yielding a function from $M_n(X) \times M_m(X) \times \{n\}$ into M is defined by the condition (Def. 21).

- (Def. 21) Let x be an element of $M_n(X) \times M_m(X) \times \{n\}$, y be an element of $M_n(X)$, and z be an element of $M_m(X)$. If $y = (x_1)_1$ and $z = (x_1)_2$, then $(f \times g)(x) = f(y) \cdot g(z)$.

In the sequel M is a non empty multiplicative magma.

One can prove the following propositions:

- (39) Let f be a function from X into M . Then there exists a function h from $M(X)$ into M such that h is multiplicative and h extends $f \cdot (\text{the 1-canonical image of } X)^{-1}$.
- (40) Let f be a function from X into M and h, g be functions from $M(X)$ into M . Suppose that
- (i) h is multiplicative,
 - (ii) h extends $f \cdot (\text{the 1-canonical image of } X)^{-1}$,
 - (iii) g is multiplicative, and

(iv) g extends $f \cdot (\text{the 1-canonical image of } X)^{-1}$.

Then $h = g$.

For simplicity, we adopt the following rules: M, N are non empty multiplicative magmas, f is a function from M into N , H is a non empty submagma of N , and R is a compatible equivalence relation of M .

We now state three propositions:

- (41) Suppose f is multiplicative and the carrier of $H = \text{rng } f$ and $R =$ the equivalence kernel of f . Then there exists a function g from M/R into H such that $f = g \cdot$ the canonical homomorphism onto cosets of R and g is bijective and multiplicative.
- (42) Let g_1, g_2 be functions from M/R into N . Suppose $g_1 \cdot$ the canonical homomorphism onto cosets of $R = g_2 \cdot$ the canonical homomorphism onto cosets of R . Then $g_1 = g_2$.
- (43) Let M be a non empty multiplicative magma. Then there exists a non empty set X and there exists a relators r of $M(X)$ such that there exists a function from $M(X)/\text{the equivalence relation of } r$ into M which is bijective and multiplicative.

Let X, Y be non empty sets and let f be a function from X into Y . The functor $\mathbf{M}(f)$ yields a function from $M(X)$ into $M(Y)$ and is defined by:

(Def. 22) $\mathbf{M}(f)$ is multiplicative and $\mathbf{M}(f)$ extends $(\text{the 1-canonical image of } Y) \cdot f \cdot (\text{the 1-canonical image of } X)^{-1}$.

Let X, Y be non empty sets and let f be a function from X into Y . One can verify that $\mathbf{M}(f)$ is multiplicative.

In the sequel f denotes a function from X into Y and g denotes a function from Y into Z .

Next we state several propositions:

- (44) $\mathbf{M}(g \cdot f) = \mathbf{M}(g) \cdot \mathbf{M}(f)$.
- (45) For every function g from X into Z such that $Y \subseteq Z$ and $f = g$ holds $\mathbf{M}(f) = \mathbf{M}(g)$.
- (46) $\mathbf{M}(\text{id}_X) = \text{id}_{\text{dom } \mathbf{M}(f)}$.
- (47) If f is one-to-one, then $\mathbf{M}(f)$ is one-to-one.
- (48) If f is onto, then $\mathbf{M}(f)$ is onto.

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