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Partial Differentiation of Real Ternary Functions

Takao Inoué
Inaba 2205, Wing-Minamikan
Nagano, Nagano, Japan

Bing Xie
Qingdao University of Science
and Technology
China

Xiquan Liang
Qingdao University of Science
and Technology
China

Summary. In this article, we shall extend the result of [19] to discuss partial differentiation of real ternary functions (refer to [8] and [16] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [7], [12], [13], [14], [1], [2], [3], [4], [5], [8], [19], [15], [9], [18], [6], [11], [10], and [17].

1. Preliminaries

For simplicity, we use the following convention: D denotes a set, x, x_0, y, y_0, z, z_0, r, s, t denote real numbers, p, a, u, u_0 denote elements of $\mathcal{R}^3$, f, f_1, f_2, f_3, g denote partial functions from $\mathcal{R}^3$ to $\mathbb{R}$, R denotes a rest, and L denotes a linear function.

One can prove the following three propositions:

(1) $\mathrm{dom}\,\mathrm{proj}(1,3) = \mathcal{R}^3$ and $\mathrm{rng}\,\mathrm{proj}(1,3) = \mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\mathrm{proj}(1,3))(\langle x, y, z \rangle) = x$.

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(2) $\operatorname{dom}\operatorname{proj}(2,3) = \mathcal{R}^3$ and $\operatorname{rng}\operatorname{proj}(2,3) = \mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\operatorname{proj}(2,3))(\langle x,y,z\rangle) = y$.

(3) $\operatorname{dom}\operatorname{proj}(3,3) = \mathcal{R}^3$ and $\operatorname{rng}\operatorname{proj}(3,3) = \mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\operatorname{proj}(3,3))(\langle x,y,z\rangle) = z$.

2. Partial Differentiation of Real Ternary Functions

One can prove the following propositions:

(4) If $u = \langle x,y,z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1, then $\mathrm{SVF1}(1,f,u)$ is differentiable in x.

(5) If $u = \langle x,y,z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2, then $\mathrm{SVF1}(2,f,u)$ is differentiable in y.

(6) If $u = \langle x,y,z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 3, then $\mathrm{SVF1}(3,f,u)$ is differentiable in z.

(7) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0,y_0,z_0\rangle$ and $\mathrm{SVF1}(1,f,u)$ is differentiable in x_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 1.

(8) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0,y_0,z_0\rangle$ and $\mathrm{SVF1}(2,f,u)$ is differentiable in y_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 2.

(9) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0,y_0,z_0\rangle$ and $\mathrm{SVF1}(3,f,u)$ is differentiable in z_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 3.

(10) Suppose $u = \langle x_0,y_0,z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom}\mathrm{SVF1}(1,f,u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1,f,u))(x) - (\mathrm{SVF1}(1,f,u))(x_0) = L(x - x_0) + R(x - x_0)$.

(11) Suppose $u = \langle x_0,y_0,z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom}\mathrm{SVF1}(2,f,u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2,f,u))(y) - (\mathrm{SVF1}(2,f,u))(y_0) = L(y - y_0) + R(y - y_0)$.

(12) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(13) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 1,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(14) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 2,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(15) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 3,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(16) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then $r = \mathrm{partdiff}(f, u, 1)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(17) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then $r = \mathrm{partdiff}(f, u, 2)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that $r = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(18) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then $r = \mathrm{partdiff}(f, u, 3)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there

exist L, R such that $r = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(19) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 1) = (\mathrm{SVF1}(1, f, u))'(x_0)$.

(20) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 2) = (\mathrm{SVF1}(2, f, u))'(y_0)$.

(21) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 3) = (\mathrm{SVF1}(3, f, u))'(z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. We say that f is partially differentiable w.r.t. 1st coordinate on D if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 1.

We say that f is partially differentiable w.r.t. 2nd coordinate on D if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 2.

We say that f is partially differentiable w.r.t. 3rd coordinate on D if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 3.

The following three propositions are true:

(22) Suppose f is partially differentiable w.r.t. 1st coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 1.

(23) Suppose f is partially differentiable w.r.t. 2nd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 2.

(24) Suppose f is partially differentiable w.r.t. 3rd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 3.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 1st coordinate on D. The functor $f^{\mathrm{1st}}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 4) $\mathrm{dom}(f^{\mathrm{1st}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\mathrm{1st}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 1)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 2nd coordinate on D. The functor $f^{\mathrm{2nd}}_{\restriction D}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined as follows:

(Def. 5) $\mathrm{dom}(f^{2\mathrm{nd}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{2\mathrm{nd}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 2)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 3rd coordinate on D. The functor $f^{3\mathrm{rd}}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 6) $\mathrm{dom}(f^{3\mathrm{rd}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{3\mathrm{rd}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 3)$.

3. Main Properties of Partial Differentiation of Real Ternary Functions

We now state a number of propositions:

(25) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(1,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c = \{(\mathrm{proj}(1,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(1, f, u_0) \cdot (h + c) - \mathrm{SVF1}(1, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 1) = \lim(h^{-1}\,(\mathrm{SVF1}(1, f, u_0) \cdot (h + c) - \mathrm{SVF1}(1, f, u_0) \cdot c))$.

(26) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(2,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c = \{(\mathrm{proj}(2,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(2, f, u_0) \cdot (h + c) - \mathrm{SVF1}(2, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 2) = \lim(h^{-1}\,(\mathrm{SVF1}(2, f, u_0) \cdot (h + c) - \mathrm{SVF1}(2, f, u_0) \cdot c))$.

(27) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(3,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c = \{(\mathrm{proj}(3,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(3, f, u_0) \cdot (h + c) - \mathrm{SVF1}(3, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 3) = \lim(h^{-1}\,(\mathrm{SVF1}(3, f, u_0) \cdot (h + c) - \mathrm{SVF1}(3, f, u_0) \cdot c))$.

(28) Suppose that
 (i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 1, and
 (ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 1.

Then $f_1\,f_2$ is partially differentiable in u_0 w.r.t. coordinate number 1.

(29) Suppose that
 (i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 2, and
 (ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 2.

Then $f_1\, f_2$ is partially differentiable in u_0 w.r.t. coordinate number 2.

(30) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 3, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 3.

Then $f_1\, f_2$ is partially differentiable in u_0 w.r.t. coordinate number 3.

(31) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then $\mathrm{SVF1}(1, f, u_0)$ is continuous in $(\mathrm{proj}(1,3))(u_0)$.

(32) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then $\mathrm{SVF1}(2, f, u_0)$ is continuous in $(\mathrm{proj}(2,3))(u_0)$.

(33) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then $\mathrm{SVF1}(3, f, u_0)$ is continuous in $(\mathrm{proj}(3,3))(u_0)$.

(34) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(35) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(36) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

4. Grads and Curl

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{grad}(f, p)$ yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 7) $\mathrm{grad}(f,p) = \mathrm{partdiff}(f,p,1) \cdot e_1 + \mathrm{partdiff}(f,p,2) \cdot e_2 + \mathrm{partdiff}(f,p,3) \cdot e_3$.

We now state several propositions:

(37) $\mathrm{grad}(f,p) = [\mathrm{partdiff}(f,p,1), \mathrm{partdiff}(f,p,2), \mathrm{partdiff}(f,p,3)]$.

(38) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(f+g, p) = \mathrm{grad}(f,p) + \mathrm{grad}(g,p)$.

(39) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.
Then $\mathrm{grad}(f - g, p) = \mathrm{grad}(f, p) - \mathrm{grad}(g, p)$.

(40) Suppose that
(i) f is partially differentiable in p w.r.t. coordinate number 1,
(ii) f is partially differentiable in p w.r.t. coordinate number 2, and
(iii) f is partially differentiable in p w.r.t. coordinate number 3.
Then $\mathrm{grad}(r\, f, p) = r \cdot \mathrm{grad}(f, p)$.

(41) Suppose that
(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and
(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.
Then $\mathrm{grad}(s\, f + t\, g, p) = s \cdot \mathrm{grad}(f, p) + t \cdot \mathrm{grad}(g, p)$.

(42) Suppose that
(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and
(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.
Then $\mathrm{grad}(s\, f - t\, g, p) = s \cdot \mathrm{grad}(f, p) - t \cdot \mathrm{grad}(g, p)$.

(43) If f is total and constant, then $\mathrm{grad}(f, p) = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

Let a be an element of $\mathcal{R}^3$. The functor $\mathrm{unitvector}\, a$ yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 8) $\mathrm{unitvector}\, a = [\frac{a(1)}{\sqrt{a(1)^{\mathbf{2}}+a(2)^{\mathbf{2}}+a(3)^{\mathbf{2}}}}, \frac{a(2)}{\sqrt{a(1)^{\mathbf{2}}+a(2)^{\mathbf{2}}+a(3)^{\mathbf{2}}}}, \frac{a(3)}{\sqrt{a(1)^{\mathbf{2}}+a(2)^{\mathbf{2}}+a(3)^{\mathbf{2}}}}]$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p, a be elements of $\mathcal{R}^3$. The functor $\mathrm{Directiondiff}(f, p, a)$ yielding a real number is defined by:

(Def. 9) $\mathrm{Directiondiff}(f, p, a) = \mathrm{partdiff}(f, p, 1) \cdot (\mathrm{unitvector}\, a)(1) + \mathrm{partdiff}(f, p, 2) \cdot (\mathrm{unitvector}\, a)(2) + \mathrm{partdiff}(f, p, 3) \cdot (\mathrm{unitvector}\, a)(3)$.

The following propositions are true:

(44) If $a = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{Directiondiff}(f, p, a) = \frac{\mathrm{partdiff}(f,p,1) \cdot x_0}{\sqrt{{x_0}^{\mathbf{2}}+{y_0}^{\mathbf{2}}+{z_0}^{\mathbf{2}}}} + \frac{\mathrm{partdiff}(f,p,2) \cdot y_0}{\sqrt{{x_0}^{\mathbf{2}}+{y_0}^{\mathbf{2}}+{z_0}^{\mathbf{2}}}} + \frac{\mathrm{partdiff}(f,p,3) \cdot z_0}{\sqrt{{x_0}^{\mathbf{2}}+{y_0}^{\mathbf{2}}+{z_0}^{\mathbf{2}}}}$.

(45) $\mathrm{Directiondiff}(f, p, a) = |(\mathrm{grad}(f, p), \mathrm{unitvector}\, a)|$.

Let f_1, f_2, f_3 be partial functions from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{curl}(f_1, f_2, f_3, p)$ yields an element of $\mathcal{R}^3$ and is defined by:

(Def. 10) $\operatorname{curl}(f_1, f_2, f_3, p) = (\operatorname{partdiff}(f_3, p, 2) - \operatorname{partdiff}(f_2, p, 3)) \cdot e_1 + (\operatorname{partdiff}(f_1, p, 3) - \operatorname{partdiff}(f_3, p, 1)) \cdot e_2 + (\operatorname{partdiff}(f_2, p, 1) - \operatorname{partdiff}(f_1, p, 2)) \cdot e_3$.

Next we state the proposition

(46) $\operatorname{curl}(f_1, f_2, f_3, p) = [\operatorname{partdiff}(f_3, p, 2) - \operatorname{partdiff}(f_2, p, 3), \operatorname{partdiff}(f_1, p, 3) - \operatorname{partdiff}(f_3, p, 1), \operatorname{partdiff}(f_2, p, 1) - \operatorname{partdiff}(f_1, p, 2)]$.

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