

# Partial Differentiation of Real Ternary Functions

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**Summary.** In this article, we shall extend the result of [19] to discuss partial differentiation of real ternary functions (refer to [8] and [16] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [7], [12], [13], [14], [1], [2], [3], [4], [5], [8], [19], [15], [9], [18], [6], [11], [10], and [17].

## 1. PRELIMINARIES

For simplicity, we use the following convention:  $D$  denotes a set,  $x, x_0, y, y_0, z, z_0, r, s, t$  denote real numbers,  $p, a, u, u_0$  denote elements of  $\mathcal{R}^3$ ,  $f, f_1, f_2, f_3, g$  denote partial functions from  $\mathcal{R}^3$  to  $\mathbb{R}$ ,  $R$  denotes a rest, and  $L$  denotes a linear function.

One can prove the following three propositions:

- (1)  $\text{dom proj}(1, 3) = \mathcal{R}^3$  and  $\text{rng proj}(1, 3) = \mathbb{R}$  and for all elements  $x, y, z$  of  $\mathbb{R}$  holds  $(\text{proj}(1, 3))(\langle x, y, z \rangle) = x$ .

- (2)  $\text{dom proj}(2, 3) = \mathcal{R}^3$  and  $\text{rng proj}(2, 3) = \mathbb{R}$  and for all elements  $x, y, z$  of  $\mathbb{R}$  holds  $(\text{proj}(2, 3))(\langle x, y, z \rangle) = y$ .
- (3)  $\text{dom proj}(3, 3) = \mathcal{R}^3$  and  $\text{rng proj}(3, 3) = \mathbb{R}$  and for all elements  $x, y, z$  of  $\mathbb{R}$  holds  $(\text{proj}(3, 3))(\langle x, y, z \rangle) = z$ .

## 2. PARTIAL DIFFERENTIATION OF REAL TERNARY FUNCTIONS

One can prove the following propositions:

- (4) If  $u = \langle x, y, z \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1, then  $\text{SVF1}(1, f, u)$  is differentiable in  $x$ .
- (5) If  $u = \langle x, y, z \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2, then  $\text{SVF1}(2, f, u)$  is differentiable in  $y$ .
- (6) If  $u = \langle x, y, z \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3, then  $\text{SVF1}(3, f, u)$  is differentiable in  $z$ .
- (7) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and  $\text{SVF1}(1, f, u)$  is differentiable in  $x_0$ ,
  - (ii)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1.
- (8) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and  $\text{SVF1}(2, f, u)$  is differentiable in  $y_0$ ,
  - (ii)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2.
- (9) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and  $\text{SVF1}(3, f, u)$  is differentiable in  $z_0$ ,
  - (ii)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3.
- (10) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1. Then there exists a neighbourhood  $N$  of  $x_0$  such that  $N \subseteq \text{dom SVF1}(1, f, u)$  and there exist  $L, R$  such that for every  $x$  such that  $x \in N$  holds  $(\text{SVF1}(1, f, u))(x) - (\text{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$ .
- (11) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2. Then there exists a neighbourhood  $N$  of  $y_0$  such that  $N \subseteq \text{dom SVF1}(2, f, u)$  and there exist  $L, R$  such that for every  $y$  such that  $y \in N$  holds  $(\text{SVF1}(2, f, u))(y) - (\text{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$ .

- (12) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3. Then there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq \text{dom SVF1}(3, f, u)$  and there exist  $L, R$  such that for every  $z$  such that  $z \in N$  holds  $(\text{SVF1}(3, f, u))(z) - (\text{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$ .
- (13) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1,
  - (ii) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $x_0$  such that  $N \subseteq \text{dom SVF1}(1, f, u)$  and there exist  $L, R$  such that for every  $x$  such that  $x \in N$  holds  $(\text{SVF1}(1, f, u))(x) - (\text{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$ .
- (14) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2,
  - (ii) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $y_0$  such that  $N \subseteq \text{dom SVF1}(2, f, u)$  and there exist  $L, R$  such that for every  $y$  such that  $y \in N$  holds  $(\text{SVF1}(2, f, u))(y) - (\text{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$ .
- (15) Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and  $u$  be an element of  $\mathcal{R}^3$ . Then the following statements are equivalent
- (i)  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3,
  - (ii) there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq \text{dom SVF1}(3, f, u)$  and there exist  $L, R$  such that for every  $z$  such that  $z \in N$  holds  $(\text{SVF1}(3, f, u))(z) - (\text{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$ .
- (16) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1. Then  $r = \text{partdiff}(f, u, 1)$  if and only if there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $x_0$  such that  $N \subseteq \text{dom SVF1}(1, f, u)$  and there exist  $L, R$  such that  $r = L(1)$  and for every  $x$  such that  $x \in N$  holds  $(\text{SVF1}(1, f, u))(x) - (\text{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$ .
- (17) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2. Then  $r = \text{partdiff}(f, u, 2)$  if and only if there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $y_0$  such that  $N \subseteq \text{dom SVF1}(2, f, u)$  and there exist  $L, R$  such that  $r = L(1)$  and for every  $y$  such that  $y \in N$  holds  $(\text{SVF1}(2, f, u))(y) - (\text{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$ .
- (18) Suppose  $u = \langle x_0, y_0, z_0 \rangle$  and  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3. Then  $r = \text{partdiff}(f, u, 3)$  if and only if there exist real numbers  $x_0, y_0, z_0$  such that  $u = \langle x_0, y_0, z_0 \rangle$  and there exists a neighbourhood  $N$  of  $z_0$  such that  $N \subseteq \text{dom SVF1}(3, f, u)$  and there

exist  $L, R$  such that  $r = L(1)$  and for every  $z$  such that  $z \in N$  holds  
 $(\text{SVF1}(3, f, u))(z) - (\text{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$ .

(19) If  $u = \langle x_0, y_0, z_0 \rangle$ , then  $\text{partdiff}(f, u, 1) = (\text{SVF1}(1, f, u))'(x_0)$ .

(20) If  $u = \langle x_0, y_0, z_0 \rangle$ , then  $\text{partdiff}(f, u, 2) = (\text{SVF1}(2, f, u))'(y_0)$ .

(21) If  $u = \langle x_0, y_0, z_0 \rangle$ , then  $\text{partdiff}(f, u, 3) = (\text{SVF1}(3, f, u))'(z_0)$ .

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $D$  be a set. We say that  $f$  is partially differentiable w.r.t. 1st coordinate on  $D$  if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i)  $D \subseteq \text{dom } f$ , and

(ii) for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f|_D$  is partially differentiable in  $u$  w.r.t. coordinate number 1.

We say that  $f$  is partially differentiable w.r.t. 2nd coordinate on  $D$  if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i)  $D \subseteq \text{dom } f$ , and

(ii) for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f|_D$  is partially differentiable in  $u$  w.r.t. coordinate number 2.

We say that  $f$  is partially differentiable w.r.t. 3rd coordinate on  $D$  if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i)  $D \subseteq \text{dom } f$ , and

(ii) for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f|_D$  is partially differentiable in  $u$  w.r.t. coordinate number 3.

The following three propositions are true:

(22) Suppose  $f$  is partially differentiable w.r.t. 1st coordinate on  $D$ . Then  $D \subseteq \text{dom } f$  and for every  $u$  such that  $u \in D$  holds  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 1.

(23) Suppose  $f$  is partially differentiable w.r.t. 2nd coordinate on  $D$ . Then  $D \subseteq \text{dom } f$  and for every  $u$  such that  $u \in D$  holds  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 2.

(24) Suppose  $f$  is partially differentiable w.r.t. 3rd coordinate on  $D$ . Then  $D \subseteq \text{dom } f$  and for every  $u$  such that  $u \in D$  holds  $f$  is partially differentiable in  $u$  w.r.t. coordinate number 3.

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $D$  be a set. Let us assume that  $f$  is partially differentiable w.r.t. 1st coordinate on  $D$ . The functor  $f|_D^{1\text{st}}$  yielding a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  is defined as follows:

(Def. 4)  $\text{dom}(f|_D^{1\text{st}}) = D$  and for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f|_D^{1\text{st}}(u) = \text{partdiff}(f, u, 1)$ .

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $D$  be a set. Let us assume that  $f$  is partially differentiable w.r.t. 2nd coordinate on  $D$ . The functor  $f|_D^{2\text{nd}}$  yields a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and is defined as follows:

(Def. 5)  $\text{dom}(f_{|D}^{2\text{nd}}) = D$  and for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f_{|D}^{2\text{nd}}(u) = \text{partdiff}(f, u, 2)$ .

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $D$  be a set. Let us assume that  $f$  is partially differentiable w.r.t. 3rd coordinate on  $D$ . The functor  $f_{|D}^{3\text{rd}}$  yielding a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  is defined as follows:

(Def. 6)  $\text{dom}(f_{|D}^{3\text{rd}}) = D$  and for every element  $u$  of  $\mathcal{R}^3$  such that  $u \in D$  holds  $f_{|D}^{3\text{rd}}(u) = \text{partdiff}(f, u, 3)$ .

### 3. MAIN PROPERTIES OF PARTIAL DIFFERENTIATION OF REAL TERNARY FUNCTIONS

We now state a number of propositions:

- (25) Let  $u_0$  be an element of  $\mathcal{R}^3$  and  $N$  be a neighbourhood of  $(\text{proj}(1, 3))(u_0)$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1 and  $N \subseteq \text{dom SVF1}(1, f, u_0)$ . Let  $h$  be a convergent to 0 sequence of real numbers and  $c$  be a constant sequence of real numbers. Suppose  $\text{rng } c = \{(\text{proj}(1, 3))(u_0)\}$  and  $\text{rng}(h + c) \subseteq N$ . Then  $h^{-1}(\text{SVF1}(1, f, u_0) \cdot (h + c) - \text{SVF1}(1, f, u_0) \cdot c)$  is convergent and  $\text{partdiff}(f, u_0, 1) = \lim(h^{-1}(\text{SVF1}(1, f, u_0) \cdot (h + c) - \text{SVF1}(1, f, u_0) \cdot c))$ .
- (26) Let  $u_0$  be an element of  $\mathcal{R}^3$  and  $N$  be a neighbourhood of  $(\text{proj}(2, 3))(u_0)$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2 and  $N \subseteq \text{dom SVF1}(2, f, u_0)$ . Let  $h$  be a convergent to 0 sequence of real numbers and  $c$  be a constant sequence of real numbers. Suppose  $\text{rng } c = \{(\text{proj}(2, 3))(u_0)\}$  and  $\text{rng}(h + c) \subseteq N$ . Then  $h^{-1}(\text{SVF1}(2, f, u_0) \cdot (h + c) - \text{SVF1}(2, f, u_0) \cdot c)$  is convergent and  $\text{partdiff}(f, u_0, 2) = \lim(h^{-1}(\text{SVF1}(2, f, u_0) \cdot (h + c) - \text{SVF1}(2, f, u_0) \cdot c))$ .
- (27) Let  $u_0$  be an element of  $\mathcal{R}^3$  and  $N$  be a neighbourhood of  $(\text{proj}(3, 3))(u_0)$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3 and  $N \subseteq \text{dom SVF1}(3, f, u_0)$ . Let  $h$  be a convergent to 0 sequence of real numbers and  $c$  be a constant sequence of real numbers. Suppose  $\text{rng } c = \{(\text{proj}(3, 3))(u_0)\}$  and  $\text{rng}(h + c) \subseteq N$ . Then  $h^{-1}(\text{SVF1}(3, f, u_0) \cdot (h + c) - \text{SVF1}(3, f, u_0) \cdot c)$  is convergent and  $\text{partdiff}(f, u_0, 3) = \lim(h^{-1}(\text{SVF1}(3, f, u_0) \cdot (h + c) - \text{SVF1}(3, f, u_0) \cdot c))$ .
- (28) Suppose that
- (i)  $f_1$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1, and
  - (ii)  $f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1.
- Then  $f_1 f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1.
- (29) Suppose that
- (i)  $f_1$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2, and
  - (ii)  $f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2.

Then  $f_1 f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2.

- (30) Suppose that
- (i)  $f_1$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3, and
  - (ii)  $f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3.
- Then  $f_1 f_2$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3.
- (31) Let  $u_0$  be an element of  $\mathcal{R}^3$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1. Then  $\text{SVF1}(1, f, u_0)$  is continuous in  $(\text{proj}(1, 3))(u_0)$ .
- (32) Let  $u_0$  be an element of  $\mathcal{R}^3$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2. Then  $\text{SVF1}(2, f, u_0)$  is continuous in  $(\text{proj}(2, 3))(u_0)$ .
- (33) Let  $u_0$  be an element of  $\mathcal{R}^3$ . Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3. Then  $\text{SVF1}(3, f, u_0)$  is continuous in  $(\text{proj}(3, 3))(u_0)$ .
- (34) Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 1. Then there exists  $R$  such that  $R(0) = 0$  and  $R$  is continuous in 0.
- (35) Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 2. Then there exists  $R$  such that  $R(0) = 0$  and  $R$  is continuous in 0.
- (36) Suppose  $f$  is partially differentiable in  $u_0$  w.r.t. coordinate number 3. Then there exists  $R$  such that  $R(0) = 0$  and  $R$  is continuous in 0.

#### 4. GRADS AND CURL

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $p$  be an element of  $\mathcal{R}^3$ . The functor  $\text{grad}(f, p)$  yields an element of  $\mathcal{R}^3$  and is defined as follows:

(Def. 7)  $\text{grad}(f, p) = \text{partdiff}(f, p, 1) \cdot e_1 + \text{partdiff}(f, p, 2) \cdot e_2 + \text{partdiff}(f, p, 3) \cdot e_3$ .

We now state several propositions:

- (37)  $\text{grad}(f, p) = [\text{partdiff}(f, p, 1), \text{partdiff}(f, p, 2), \text{partdiff}(f, p, 3)]$ .
- (38) Suppose that
- (i)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3, and
  - (ii)  $g$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3.
- Then  $\text{grad}(f + g, p) = \text{grad}(f, p) + \text{grad}(g, p)$ .
- (39) Suppose that
- (i)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3, and

- (ii)  $g$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3.

Then  $\text{grad}(f - g, p) = \text{grad}(f, p) - \text{grad}(g, p)$ .

(40) Suppose that

- (i)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 1,
- (ii)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 2, and
- (iii)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 3.

Then  $\text{grad}(r f, p) = r \cdot \text{grad}(f, p)$ .

(41) Suppose that

- (i)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3, and
- (ii)  $g$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3.

Then  $\text{grad}(s f + t g, p) = s \cdot \text{grad}(f, p) + t \cdot \text{grad}(g, p)$ .

(42) Suppose that

- (i)  $f$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3, and
- (ii)  $g$  is partially differentiable in  $p$  w.r.t. coordinate number 1, partially differentiable in  $p$  w.r.t. coordinate number 2, and partially differentiable in  $p$  w.r.t. coordinate number 3.

Then  $\text{grad}(s f - t g, p) = s \cdot \text{grad}(f, p) - t \cdot \text{grad}(g, p)$ .

(43) If  $f$  is total and constant, then  $\text{grad}(f, p) = 0_{\mathcal{E}_T^3}$ .

Let  $a$  be an element of  $\mathcal{R}^3$ . The functor unitvector  $a$  yields an element of  $\mathcal{R}^3$  and is defined as follows:

(Def. 8) unitvector  $a = \left[ \frac{a(1)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}}, \frac{a(2)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}}, \frac{a(3)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}} \right]$ .

Let  $f$  be a partial function from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $p, a$  be elements of  $\mathcal{R}^3$ .

The functor Directiondiff( $f, p, a$ ) yielding a real number is defined by:

(Def. 9) Directiondiff( $f, p, a$ ) = partdiff( $f, p, 1$ ) · (unitvector  $a$ )(1) + partdiff( $f, p, 2$ ) · (unitvector  $a$ )(2) + partdiff( $f, p, 3$ ) · (unitvector  $a$ )(3).

The following propositions are true:

(44) If  $a = \langle x_0, y_0, z_0 \rangle$ , then Directiondiff( $f, p, a$ ) =  $\frac{\text{partdiff}(f,p,1) \cdot x_0}{\sqrt{x_0^2+y_0^2+z_0^2}} + \frac{\text{partdiff}(f,p,2) \cdot y_0}{\sqrt{x_0^2+y_0^2+z_0^2}} + \frac{\text{partdiff}(f,p,3) \cdot z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}$ .

(45) Directiondiff( $f, p, a$ ) = |(grad( $f, p$ ), unitvector  $a$ )|.

Let  $f_1, f_2, f_3$  be partial functions from  $\mathcal{R}^3$  to  $\mathbb{R}$  and let  $p$  be an element of  $\mathcal{R}^3$ . The functor curl( $f_1, f_2, f_3, p$ ) yields an element of  $\mathcal{R}^3$  and is defined by:

$$\text{(Def. 10)} \quad \text{curl}(f_1, f_2, f_3, p) = (\text{partdiff}(f_3, p, 2) - \text{partdiff}(f_2, p, 3)) \cdot e_1 + \\ (\text{partdiff}(f_1, p, 3) - \text{partdiff}(f_3, p, 1)) \cdot e_2 + (\text{partdiff}(f_2, p, 1) - \\ \text{partdiff}(f_1, p, 2)) \cdot e_3.$$

Next we state the proposition

$$\text{(46)} \quad \text{curl}(f_1, f_2, f_3, p) = [\text{partdiff}(f_3, p, 2) - \text{partdiff}(f_2, p, 3), \text{partdiff}(f_1, p, 3) - \\ \text{partdiff}(f_3, p, 1), \text{partdiff}(f_2, p, 1) - \text{partdiff}(f_1, p, 2)].$$

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