

Fixpoint Theorem for Continuous Functions on Chain-Complete Posets

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Summary. This text includes the definition of chain-complete poset, fixpoint theorem on it, and the definition of the function space of continuous functions on chain-complete posets [10].

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The papers [8], [4], [5], [3], [1], [9], [7], [11], [13], [12], [2], [14], and [6] provide the notation and terminology for this paper.

1. MONOTONE FUNCTIONS, CHAIN AND CHAIN-COMPLETE POSETS

Let P be a non empty poset. Observe that there exists a chain of P which is non empty.

Let I_1 be a relational structure. We say that I_1 is chain-complete if and only if:

- (Def. 1) I_1 is lower-bounded and for every chain L of I_1 such that L is non empty holds $\sup L$ exists in I_1 .

One can prove the following proposition

- (1) Let P_1, P_2 be non empty posets, K be a non empty chain of P_1 , and h be a monotone function from P_1 into P_2 . Then $h \circ K$ is a non empty chain of P_2 .

Let us note that there exists a poset which is strict, chain-complete, and non empty.

Let us mention that every relational structure which is chain-complete is also lower-bounded.

For simplicity, we adopt the following rules: x, y denote sets, P, Q denote strict chain-complete non empty posets, L denotes a non empty chain of P , M denotes a non empty chain of Q , p denotes an element of P , f denotes a monotone function from P into Q , and g, g_1, g_2 denote monotone functions from P into P .

We now state the proposition

$$(2) \quad \sup(f \circ L) \leq f(\sup L).$$

2. FIXPOINT THEOREM FOR CONTINUOUS FUNCTIONS ON CHAIN-COMPLETE POSETS

Let P be a non empty poset, let g be a monotone function from P into P , and let p be an element of P . The functor $\text{iterSet}(g, p)$ yields a non empty set and is defined by:

$$(\text{Def. 2}) \quad \text{iterSet}(g, p) = \{x \in P: \forall_{n: \text{natural number}} x = g^n(p)\}.$$

Next we state the proposition

$$(3) \quad \text{iterSet}(g, \perp_P) \text{ is a non empty chain of } P.$$

Let us consider P and let g be a monotone function from P into P . The functor $\text{iter-min } g$ yields a non empty chain of P and is defined by:

$$(\text{Def. 3}) \quad \text{iter-min } g = \text{iterSet}(g, \perp_P).$$

The following propositions are true:

$$(4) \quad \sup \text{iter-min } g = \sup(g \circ \text{iter-min } g).$$

$$(5) \quad \text{If } g_1 \leq g_2, \text{ then } \sup \text{iter-min } g_1 \leq \sup \text{iter-min } g_2.$$

Let P, Q be non empty posets and let f be a function from P into Q . We say that f is continuous if and only if:

$$(\text{Def. 4}) \quad f \text{ is monotone and for every non empty chain } L \text{ of } P \text{ holds } f \text{ preserves } \sup \text{ of } L.$$

We now state two propositions:

$$(6) \quad \text{For every function } f \text{ from } P \text{ into } Q \text{ holds } f \text{ is continuous iff } f \text{ is monotone and for every } L \text{ holds } f(\sup L) = \sup(f \circ L).$$

$$(7) \quad \text{For every element } z \text{ of } Q \text{ holds } P \mapsto z \text{ is continuous.}$$

Let us consider P, Q . Observe that there exists a function from P into Q which is continuous.

Let us consider P, Q . One can verify that every function from P into Q which is continuous is also monotone.

The following proposition is true

$$(8) \quad \text{For every monotone function } f \text{ from } P \text{ into } Q \text{ such that for every } L \text{ holds } f(\sup L) \leq \sup(f \circ L) \text{ holds } f \text{ is continuous.}$$

Let us consider P and let g be a monotone function from P into P . Let us assume that g is continuous. The least fixpoint of g yields an element of P and is defined by the conditions (Def. 5).

- (Def. 5)(i) The least fixpoint of g is a fixpoint of g , and
(ii) for every p such that p is a fixpoint of g holds the least fixpoint of $g \leq p$.

One can prove the following propositions:

- (9) For every continuous function g from P into P holds the least fixpoint of $g = \text{sup iter-min } g$.
(10) Let g_1, g_2 be continuous functions from P into P . If $g_1 \leq g_2$, then the least fixpoint of $g_1 \leq$ the least fixpoint of g_2 .

3. FUNCTION SPACE OF CONTINUOUS FUNCTIONS ON CHAIN-COMPLETE POSETS

Let us consider P, Q . The functor $\text{ConFuncs}(P, Q)$ yields a non empty set and is defined by the condition (Def. 6).

- (Def. 6) $\text{ConFuncs}(P, Q) = \{x; x \text{ ranges over elements of (the carrier of } Q)\}^{\text{the carrier of } P: \bigvee_{f: \text{continuous function from } P \text{ into } Q} f = x}$.

Let us consider P, Q . The functor $\text{ConRelat}(P, Q)$ yielding a binary relation on $\text{ConFuncs}(P, Q)$ is defined by the condition (Def. 7).

- (Def. 7) Let given x, y . Then $\langle x, y \rangle \in \text{ConRelat}(P, Q)$ if and only if the following conditions are satisfied:
(i) $x \in \text{ConFuncs}(P, Q)$,
(ii) $y \in \text{ConFuncs}(P, Q)$, and
(iii) there exist functions f, g from P into Q such that $x = f$ and $y = g$ and $f \leq g$.

Let us consider P, Q . One can verify the following observations:

- * $\text{ConRelat}(P, Q)$ is reflexive,
- * $\text{ConRelat}(P, Q)$ is transitive, and
- * $\text{ConRelat}(P, Q)$ is antisymmetric.

Let us consider P, Q . The functor $\text{ConPoset}(P, Q)$ yielding a strict non empty poset is defined as follows:

- (Def. 8) $\text{ConPoset}(P, Q) = \langle \text{ConFuncs}(P, Q), \text{ConRelat}(P, Q) \rangle$.

In the sequel F is a non empty chain of $\text{ConPoset}(P, Q)$.

Let us consider P, Q, F, p . The functor $F\text{-image}(p)$ yielding a non empty chain of Q is defined as follows:

- (Def. 9) $F\text{-image}(p) = \{x \in Q: \bigvee_{f: \text{continuous function from } P \text{ into } Q} (f \in F \wedge x = f(p))\}$.

Let us consider P, Q, F . The functor $\text{sup-func } F$ yields a function from P into Q and is defined as follows:

(Def. 10) For all p, M such that $M = F\text{-image}(p)$ holds $(\text{sup-func } F)(p) = \text{sup } M$.

Let us consider P, Q, F . One can check that $\text{sup-func } F$ is continuous.

The following proposition is true

(11) $\text{Sup } F$ exists in $\text{ConPoset}(P, Q)$ and $\text{sup-func } F = \bigsqcup_{\text{ConPoset}(P, Q)} F$.

Let us consider P, Q . The functor $\text{min-func}(P, Q)$ yielding a function from P into Q is defined as follows:

(Def. 11) $\text{min-func}(P, Q) = P \mapsto \perp_Q$.

Let us consider P, Q . One can check that $\text{min-func}(P, Q)$ is continuous.

The following proposition is true

(12) For every element f of $\text{ConPoset}(P, Q)$ such that $f = \text{min-func}(P, Q)$ holds $f \leq$ the carrier of $\text{ConPoset}(P, Q)$.

Let us consider P, Q . Note that $\text{ConPoset}(P, Q)$ is chain-complete.

4. CONTINUITY OF FIXPOINT FUNCTION FROM $\text{CONPOSET}(P, P)$ INTO P

Let us consider P . The functor $\text{fix-func } P$ yielding a function from $\text{ConPoset}(P, P)$ into P is defined by the condition (Def. 12).

(Def. 12) Let g be an element of $\text{ConPoset}(P, P)$ and h be a continuous function from P into P . If $g = h$, then $(\text{fix-func } P)(g) =$ the least fixpoint of h .

Let us consider P . One can check that $\text{fix-func } P$ is continuous.

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