

Nilpotent Groups

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Summary. This article describes the concept of the nilpotent group and some properties of the nilpotent groups.

MML identifier: GRNILP_1, version: 7.11.04 4.130.1076

The papers [2], [3], [4], [6], [7], [5], [8], [9], [10], and [1] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: x denotes a set, G denotes a group, A, B, H, H_1, H_2 denote subgroups of G , a, b, c denote elements of G , F denotes a finite sequence of elements of the carrier of G , and i, j denote elements of \mathbb{N} .

One can prove the following propositions:

- (1) $a^b = a \cdot [a, b]$.
- (2) $[a, b]^{-1} = [a, b^{-1}]^b$.
- (3) $[a, b]^{-1} = [a^{-1}, b]^a$.
- (4) $([a, b^{-1}]^b)^{-1} = [b^{-1}, a]^b$.
- (5) $[a, b^{-1}, c]^b = [[a, b^{-1}]^b, c^b]$.
- (6) $[a, b^{-1}]^b = [b, a]$.
- (7) $[a, b^{-1}, c]^b = [b, a, c^b]$.
- (8) $[a, b, c^a] \cdot [c, a, b^c] \cdot [b, c, a^b] = \mathbf{1}_G$.

(9) $[A, B]$ is a subgroup of $[B, A]$.

(10) $[A, B] = [B, A]$.

Let us consider G, A, B . Let us note that the functor $[A, B]$ is commutative.

One can prove the following propositions:

(11) If B is a subgroup of A , then the commutators of A & $B \subseteq \overline{A}$.

(12) If B is a subgroup of A , then $[A, B]$ is a subgroup of A .

(13) If B is a subgroup of A , then $[B, A]$ is a subgroup of A .

(14) If $[H_1, \Omega_G]$ is a subgroup of H_2 , then $[H_1 \cap H, H]$ is a subgroup of $H_2 \cap H$.

(15) $[H_1, H_2]$ is a subgroup of $[H_1, \Omega_G]$.

(16) A is a normal subgroup of G iff $[A, \Omega_G]$ is a subgroup of A .

Let us consider G . The normal subgroups of G yields a set and is defined by:

(Def. 1) $x \in$ the normal subgroups of G iff x is a strict normal subgroup of G .

Let us consider G . One can verify that the normal subgroups of G is non empty.

Next we state three propositions:

(17) Let F be a finite sequence of elements of the normal subgroups of G and given j . If $j \in \text{dom } F$, then $F(j)$ is a strict normal subgroup of G .

(18) The normal subgroups of $G \subseteq \text{SubGr } G$.

(19) Every finite sequence of elements of the normal subgroups of G is a finite sequence of elements of $\text{SubGr } G$.

Let I_1 be a group. We say that I_1 is nilpotent if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a finite sequence F of elements of the normal subgroups of I_1 such that

(i) $\text{len } F > 0$,

(ii) $F(1) = \Omega_{(I_1)}$,

(iii) $F(\text{len } F) = \{\mathbf{1}\}_{(I_1)}$, and

(iv) for every i such that $i, i+1 \in \text{dom } F$ and for all strict normal subgroups G_1, G_2 of I_1 such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and $G_1 /_{(G_2)(G_1)}$ is a subgroup of $Z(I_1 /_{G_2})$.

Let us note that there exists a group which is nilpotent and strict.

We now state four propositions:

(20) Let G_1 be a subgroup of G and N be a strict normal subgroup of G . Suppose N is a subgroup of G_1 and $G_1 /_{(N)(G_1)}$ is a subgroup of $Z(G /_N)$. Then $[G_1, \Omega_G]$ is a subgroup of N .

(21) Let G_1 be a subgroup of G and N be a normal subgroup of G . Suppose N is a strict subgroup of G_1 and $[G_1, \Omega_G]$ is a strict subgroup of N . Then $G_1 /_{(N)(G_1)}$ is a subgroup of $Z(G /_N)$.

- (22) Let G be a group. Then G is nilpotent if and only if there exists a finite sequence F of elements of the normal subgroups of G such that $\text{len } F > 0$ and $F(1) = \Omega_G$ and $F(\text{len } F) = \{\mathbf{1}\}_G$ and for every i such that $i, i+1 \in \text{dom } F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and $[G_1, \Omega_G]$ is a subgroup of G_2 .
- (23) Let G be a group, H, G_1 be subgroups of G , G_2 be a strict normal subgroup of G , H_1 be a subgroup of H , and H_2 be a normal subgroup of H . Suppose G_2 is a subgroup of G_1 and $G_1/(G_2)_{(G_1)}$ is a subgroup of $Z(G/G_2)$ and $H_1 = G_1 \cap H$ and $H_2 = G_2 \cap H$. Then $H_1/(H_2)_{(H_1)}$ is a subgroup of $Z(H/H_2)$.

Let G be a nilpotent group. Note that every subgroup of G is nilpotent.

Let us mention that every group which is commutative is also nilpotent and every group which is cyclic is also nilpotent.

We now state four propositions:

- (24) Let G, H be strict groups, h be a homomorphism from G to H , A be a strict subgroup of G , and a, b be elements of G . Then $h(a) \cdot h(b) \cdot h^\circ A = h^\circ(a \cdot b \cdot A)$ and $h^\circ A \cdot h(a) \cdot h(b) = h^\circ(A \cdot a \cdot b)$.
- (25) Let G, H be strict groups, h be a homomorphism from G to H , A be a strict subgroup of G , a, b be elements of G , H_1 be a subgroup of $\text{Im } h$, and a_1, b_1 be elements of $\text{Im } h$. If $a_1 = h(a)$ and $b_1 = h(b)$ and $H_1 = h^\circ A$, then $a_1 \cdot b_1 \cdot H_1 = h(a) \cdot h(b) \cdot h^\circ A$.
- (26) Let G, H be strict groups, h be a homomorphism from G to H , G_1 be a strict subgroup of G , G_2 be a strict normal subgroup of G , H_1 be a strict subgroup of $\text{Im } h$, and H_2 be a strict normal subgroup of $\text{Im } h$. Suppose G_2 is a strict subgroup of G_1 and $G_1/(G_2)_{(G_1)}$ is a subgroup of $Z(G/G_2)$ and $H_1 = h^\circ G_1$ and $H_2 = h^\circ G_2$. Then $H_1/(H_2)_{(H_1)}$ is a subgroup of $Z(\text{Im } h/H_2)$.
- (27) Let G, H be strict groups, h be a homomorphism from G to H , and A be a strict normal subgroup of G . Then $h^\circ A$ is a strict normal subgroup of $\text{Im } h$.

Let G be a strict nilpotent group, let H be a strict group, and let h be a homomorphism from G to H . One can check that $\text{Im } h$ is nilpotent.

Let G be a strict nilpotent group and let N be a strict normal subgroup of G . Note that G/N is nilpotent.

One can prove the following three propositions:

- (28) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
- (i) $\text{len } F > 0$,
 - (ii) $F(1) = \Omega_G$,
 - (iii) $F(\text{len } F) = \{\mathbf{1}\}_G$, and

- (iv) for every i such that $i, i + 1 \in \text{dom } F$ and for every strict normal subgroup G_1 of G such that $G_1 = F(i)$ holds $[G_1, \Omega_G] = F(i + 1)$.
Then G is nilpotent.
- (29) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
- (i) $\text{len } F > 0$,
 - (ii) $F(1) = \Omega_G$,
 - (iii) $F(\text{len } F) = \{\mathbf{1}\}_G$, and
 - (iv) for every i such that $i, i + 1 \in \text{dom } F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i + 1)$ holds G_2 is a subgroup of G_1 and G/G_2 is a commutative group.
Then G is nilpotent.
- (30) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
- (i) $\text{len } F > 0$,
 - (ii) $F(1) = \Omega_G$,
 - (iii) $F(\text{len } F) = \{\mathbf{1}\}_G$, and
 - (iv) for every i such that $i, i + 1 \in \text{dom } F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i + 1)$ holds G_2 is a subgroup of G_1 and G/G_2 is a cyclic group.
Then G is nilpotent.

Let us mention that every group which is nilpotent is also solvable.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(5):955–962, 1990.
- [6] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [7] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(5):855–864, 1990.
- [8] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(4):461–466, 1991.
- [9] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(4):573–578, 1991.
- [10] Katarzyna Zawadzka. Solvable groups. *Formalized Mathematics*, 5(1):145–147, 1996.

Received November 10, 2009