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Difference and Difference Quotient. Part III

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Summary. In this article, we give some important theorems of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions.

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The terminology and notation used in this paper have been introduced in the following papers: [6], [2], [1], [4], [11], [7], [5], [8], [12], [9], [10], and [3].

We follow the rules: n, m are elements of $\mathbb{N}$, h, k, r, r_1, r_2, x, x_0, x_1, x_2, x_3 are real numbers, and f, f_1, f_2 are functions from $\mathbb{R}$ into $\mathbb{R}$.

Next we state a number of propositions:

(1) $(\delta_h[f])(x) = (\Delta_{\frac{h}{2}}[f])(x) - (\Delta_{-\frac{h}{2}}[f])(x)$.

(2) $(\Delta_{-\frac{h}{2}}[f])(x) = -(\nabla_{\frac{h}{2}}[f])(x)$.

(3) $(\delta_h[f])(x) = (\nabla_{\frac{h}{2}}[f])(x) - (\nabla_{-\frac{h}{2}}[f])(x)$.

(4) $(\vec{\Delta}_h[r\, f_1 + f_2])(n+1)(x) = r \cdot (\vec{\Delta}_h[f_1])(n+1)(x) + (\vec{\Delta}_h[f_2])(n+1)(x)$.

(5) $(\vec{\Delta}_h[f_1 + r\, f_2])(n+1)(x) = (\vec{\Delta}_h[f_1])(n+1)(x) + r \cdot (\vec{\Delta}_h[f_2])(n+1)(x)$.

(6) $(\vec{\Delta}_h[r_1\, f_1 - r_2\, f_2])(n+1)(x) = r_1 \cdot (\vec{\Delta}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\Delta}_h[f_2])(n+1)(x)$.

(7) $(\vec{\Delta}_h[f])(1) = \Delta_h[f]$.

(8) $(\vec{\nabla}_h[r\, f_1 + f_2])(n+1)(x) = r \cdot (\vec{\nabla}_h[f_1])(n+1)(x) + (\vec{\nabla}_h[f_2])(n+1)(x)$.

(9) $(\vec{\nabla}_h[f_1 + r\, f_2])(n+1)(x) = (\vec{\nabla}_h[f_1])(n+1)(x) + r \cdot (\vec{\nabla}_h[f_2])(n+1)(x)$.

(10) $(\vec{\nabla}_h[r_1\, f_1 - r_2\, f_2])(n+1)(x) = r_1 \cdot (\vec{\nabla}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\nabla}_h[f_2])(n+1)(x)$.

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(11) $(\vec{\nabla}_h[f])(1) = \nabla_h[f]$.

(12) $(\vec{\nabla}_h[(\vec{\nabla}_h[f])(m)])(n)(x) = (\vec{\nabla}_h[f])(m+n)(x)$.

(13) $(\vec{\delta}_h[r\, f_1 + f_2])(n+1)(x) = r \cdot (\vec{\delta}_h[f_1])(n+1)(x) + (\vec{\delta}_h[f_2])(n+1)(x)$.

(14) $(\vec{\delta}_h[f_1 + r\, f_2])(n+1)(x) = (\vec{\delta}_h[f_1])(n+1)(x) + r \cdot (\vec{\delta}_h[f_2])(n+1)(x)$.

(15) $(\vec{\delta}_h[r_1\, f_1 - r_2\, f_2])(n+1)(x) = r_1 \cdot (\vec{\delta}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\delta}_h[f_2])(n+1)(x)$.

(16) $(\vec{\delta}_h[f])(1) = \delta_h[f]$.

(17) $(\vec{\delta}_h[(\vec{\delta}_h[f])(m)])(n)(x) = (\vec{\delta}_h[f])(m+n)(x)$.

(18) If $(\vec{\Delta}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x + \frac{n}{2} \cdot h)$, then $(\vec{\nabla}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x - \frac{n}{2} \cdot h)$.

(19) If $(\vec{\Delta}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x + \frac{n-1}{2} \cdot h + \frac{h}{2})$, then $(\vec{\nabla}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x - \frac{n-1}{2} \cdot h - \frac{h}{2})$.

(20) $\Delta[f](x, x+h) = \frac{(\Delta_h[f])(x)}{h}$.

(21) $\Delta[f](x-h, x) = \frac{(\nabla_h[f])(x)}{h}$.

(22) $\Delta[f](x - \frac{h}{2}, x + \frac{h}{2}) = \frac{(\delta_h[f])(x)}{h}$.

(23) $\Delta[f](x - \frac{h}{2}, x + \frac{h}{2}) = \frac{(\vec{\delta}_h[f])(1)(x)}{h}$.

(24) If $h \neq 0$, then $\Delta[f](x-h, x, x+h) = \frac{(\vec{\delta}_h[f])(2)(x)}{2 \cdot h \cdot h}$.

(25) $\Delta[f_1 - f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) - \Delta[f_2](x_0, x_1)$.

(26) $\Delta[r\, f_1 + f_2](x_0, x_1) = r \cdot \Delta[f_1](x_0, x_1) + \Delta[f_2](x_0, x_1)$.

(27) $\Delta[r\, f_1 - f_2](x_0, x_1) = r \cdot \Delta[f_1](x_0, x_1) - \Delta[f_2](x_0, x_1)$.

(28) $\Delta[f_1 + r\, f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) + r \cdot \Delta[f_2](x_0, x_1)$.

(29) $\Delta[f_1 - r\, f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) - r \cdot \Delta[f_2](x_0, x_1)$.

(30) $\Delta[r_1\, f_1 - r_2\, f_2](x_0, x_1) = r_1 \cdot \Delta[f_1](x_0, x_1) - r_2 \cdot \Delta[f_2](x_0, x_1)$.

(31) $(\vec{\nabla}_h[f_1\, f_2])(1)(x) = f_1(x) \cdot (\vec{\nabla}_h[f_2])(1)(x) + f_2(x-h) \cdot (\vec{\nabla}_h[f_1])(1)(x)$.

(32) If x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = \Delta[f](x_0, x_2, x_1)$.

In the sequel S is a sequence of real sequences.

We now state a number of propositions:

(33) Suppose that for all natural numbers n, i such that $i \leq n$ holds $S(n)(i) = \binom{n}{i} \cdot (\vec{\nabla}_h[f_1])(i)(x) \cdot (\vec{\nabla}_h[f_2])(n -' i)(x - i \cdot h)$. Then $(\vec{\nabla}_h[f_1\, f_2])(1)(x) = \sum_{\kappa=0}^{1} S(1)(\kappa)$ and $(\vec{\nabla}_h[f_1\, f_2])(2)(x) = \sum_{\kappa=0}^{2} S(2)(\kappa)$.

(34) $(\vec{\delta}_h[f_1\, f_2])(1)(x) = f_1(x + \frac{h}{2}) \cdot (\vec{\delta}_h[f_2])(1)(x) + f_2(x - \frac{h}{2}) \cdot (\vec{\delta}_h[f_1])(1)(x)$.

(35) Suppose that for all natural numbers n, i such that $i \leq n$ holds $S(n)(i) = \binom{n}{i} \cdot (\vec{\delta}_h[f_1])(i)(x + (n -' i) \cdot \frac{h}{2}) \cdot (\vec{\delta}_h[f_2])(n -' i)(x - i \cdot \frac{h}{2})$. Then $(\vec{\delta}_h[f_1\, f_2])(1)(x) = \sum_{\kappa=0}^{1} S(1)(\kappa)$ and $(\vec{\delta}_h[f_1\, f_2])(2)(x) = \sum_{\kappa=0}^{2} S(2)(\kappa)$.

(36) If for every x holds $f(x) = \sqrt{x}$ and $x_0 \neq x_1$ and $x_0 > 0$ and $x_1 > 0$, then $\Delta[f](x_0, x_1) = \frac{1}{\sqrt{x_0} + \sqrt{x_1}}$.

(37) Suppose for every x holds $f(x) = \sqrt{x}$ and x_0, x_1, x_2 are mutually different and $x_0 > 0$ and $x_1 > 0$ and $x_2 > 0$. Then $\Delta[f](x_0, x_1, x_2) = -\frac{1}{(\sqrt{x_0}+\sqrt{x_1})\cdot(\sqrt{x_0}+\sqrt{x_2})\cdot(\sqrt{x_1}+\sqrt{x_2})}$.

(38) Suppose for every x holds $f(x) = \sqrt{x}$ and x_0, x_1, x_2, x_3 are mutually different and $x_0 > 0$ and $x_1 > 0$ and $x_2 > 0$ and $x_3 > 0$.
Then $\Delta[f](x_0, x_1, x_2, x_3) = \frac{\sqrt{x_0}+\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}}{(\sqrt{x_0}+\sqrt{x_1})\cdot(\sqrt{x_0}+\sqrt{x_2})\cdot(\sqrt{x_0}+\sqrt{x_3})\cdot(\sqrt{x_1}+\sqrt{x_2})\cdot(\sqrt{x_1}+\sqrt{x_3})\cdot(\sqrt{x_2}+\sqrt{x_3})}$.

(39) If for every x holds $f(x) = \sqrt{x}$ and $x > 0$ and $x + h > 0$, then $(\Delta_h[f])(x) = \sqrt{x+h} - \sqrt{x}$.

(40) If for every x holds $f(x) = \sqrt{x}$ and $x > 0$ and $x - h > 0$, then $(\nabla_h[f])(x) = \sqrt{x} - \sqrt{x-h}$.

(41) If for every x holds $f(x) = \sqrt{x}$ and $x + \frac{h}{2} > 0$ and $x - \frac{h}{2} > 0$, then $(\delta_h[f])(x) = \sqrt{x+\frac{h}{2}} - \sqrt{x-\frac{h}{2}}$.

(42) If for every x holds $f(x) = x^{\mathbf{2}}$ and $x_0 \neq x_1$, then $\Delta[f](x_0, x_1) = x_0 + x_1$.

(43) If for every x holds $f(x) = x^{\mathbf{2}}$ and x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = 1$.

(44) If for every x holds $f(x) = x^{\mathbf{2}}$ and x_0, x_1, x_2, x_3 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3) = 0$.

(45) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\Delta_h[f])(x) = 2 \cdot x \cdot h + h^{\mathbf{2}}$.

(46) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\nabla_h[f])(x) = h \cdot (2 \cdot x - h)$.

(47) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\delta_h[f])(x) = 2 \cdot h \cdot x$.

(48) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x_0 \neq x_1$ and $x_0 \neq 0$ and $x_1 \neq 0$, then $\Delta[f](x_0, x_1) = -\frac{k}{x_0 \cdot x_1} \cdot (\frac{1}{x_0} + \frac{1}{x_1})$.

(49) Suppose for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and x_0, x_1, x_2 are mutually different. Then $\Delta[f](x_0, x_1, x_2) = \frac{k}{x_0 \cdot x_1 \cdot x_2} \cdot (\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2})$.

(50) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x \neq 0$ and $x+h \neq 0$, then $(\Delta_h[f])(x) = \frac{(-k)\cdot h\cdot(2\cdot x+h)}{(x^{\mathbf{2}}+h\cdot x)^{\mathbf{2}}}$.

(51) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x \neq 0$ and $x-h \neq 0$, then $(\nabla_h[f])(x) = \frac{(-k)\cdot h\cdot(2\cdot x-h)}{(x^{\mathbf{2}}-x\cdot h)^{\mathbf{2}}}$.

(52) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x + \frac{h}{2} \neq 0$ and $x - \frac{h}{2} \neq 0$, then $(\delta_h[f])(x) = \frac{-2\cdot h\cdot k\cdot x}{(x^{\mathbf{2}}-(\frac{h}{2})^{\mathbf{2}})^{\mathbf{2}}}$.

(53) Δ[(the function sin) (the function sin) (the function sin)]$(x_0, x_1) = \frac{\frac{1}{2}\cdot(3\cdot\cos(\frac{x_0+x_1}{2})\cdot\sin(\frac{x_0-x_1}{2})-\cos(\frac{3\cdot(x_0+x_1)}{2})\cdot\sin(\frac{3\cdot(x_0-x_1)}{2}))}{x_0-x_1}$.

(54) $(\Delta_h$[(the function sin) (the function sin) (the function sin)]$)(x) = \frac{1}{2} \cdot (3 \cdot \cos(\frac{2\cdot x+h}{2}) \cdot \sin(\frac{h}{2}) - \cos(\frac{3\cdot(2\cdot x+h)}{2}) \cdot \sin(\frac{3\cdot h}{2}))$.

(55) $(\nabla_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (3 \cdot \cos(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}) - \cos(\frac{3 \cdot (2 \cdot x - h)}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$.

(56) $(\delta_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (3 \cdot \cos x \cdot \sin(\frac{h}{2}) - \cos(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$.

(57) $\Delta[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})](x_0, x_1) = -\frac{\frac{1}{2} \cdot (3 \cdot \sin(\frac{x_0+x_1}{2}) \cdot \sin(\frac{x_0-x_1}{2}) + \sin(\frac{3 \cdot x_0 + 3 \cdot x_1}{2}) \cdot \sin(\frac{3 \cdot x_0 - 3 \cdot x_1}{2}))}{x_0 - x_1}$.

(58) $(\Delta_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{h}{2}) + \sin(\frac{3 \cdot (2 \cdot x + h)}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$.

(59) $(\nabla_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}) + \sin(\frac{3 \cdot (2 \cdot x - h)}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$.

(60) $(\delta_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin x \cdot \sin(\frac{h}{2}) + \sin(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$.

(61) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x_0 \neq 0$ and $\sin x_1 \neq 0$, then $\Delta[f](x_0, x_1) = -\frac{\frac{2 \cdot (\sin x_1 - \sin x_0)}{\cos(x_0+x_1) - \cos(x_0-x_1)}}{x_0 - x_1}$.

(62) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x \neq 0$ and $\sin(x+h) \neq 0$, then $(\Delta_h[f])(x) = -\frac{2 \cdot (\sin x - \sin(x+h))}{\cos(2 \cdot x + h) - \cos h}$.

(63) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x \neq 0$ and $\sin(x-h) \neq 0$, then $(\nabla_h[f])(x) = \frac{(-2) \cdot (\sin(x-h) - \sin x)}{\cos(2 \cdot x - h) - \cos h}$.

(64) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin(x + \frac{h}{2}) \neq 0$ and $\sin(x - \frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = -\frac{2 \cdot (\sin(x - \frac{h}{2}) - \sin(x + \frac{h}{2}))}{\cos(2 \cdot x) - \cos h}$.

(65) If for every x holds $f(x) = \frac{1}{\cos x}$ and $x_0 \neq x_1$ and $\cos x_0 \neq 0$ and $\cos x_1 \neq 0$, then $\Delta[f](x_0, x_1) = \frac{\frac{2 \cdot (\cos x_1 - \cos x_0)}{\cos(x_0+x_1) + \cos(x_0-x_1)}}{x_0 - x_1}$.

(66) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos x \neq 0$ and $\cos(x+h) \neq 0$, then $(\Delta_h[f])(x) = \frac{2 \cdot (\cos x - \cos(x+h))}{\cos(2 \cdot x + h) + \cos h}$.

(67) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos x \neq 0$ and $\cos(x-h) \neq 0$, then $(\nabla_h[f])(x) = \frac{2 \cdot (\cos(x-h) - \cos x)}{\cos(2 \cdot x - h) + \cos h}$.

(68) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos(x + \frac{h}{2}) \neq 0$ and $\cos(x - \frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{2 \cdot (\cos(x - \frac{h}{2}) - \cos(x + \frac{h}{2}))}{\cos(2 \cdot x) + \cos h}$.

(69) Suppose for every x holds $f(x) = \frac{1}{(\sin x)^2}$ and $x_0 \neq x_1$ and $\sin x_0 \neq 0$ and $\sin x_1 \neq 0$. Then $\Delta[f](x_0, x_1) = \frac{16 \cdot \cos(\frac{x_1+x_0}{2}) \cdot \sin(\frac{x_1-x_0}{2}) \cdot \cos(\frac{x_1-x_0}{2}) \cdot \sin(\frac{x_1+x_0}{2})}{(\cos(x_0+x_1) - \cos(x_0-x_1))^2 \cdot (x_0 - x_1)}$.

(70) If for every x holds $f(x) = \frac{1}{(\sin x)^2}$ and $\sin x \neq 0$ and $\sin(x+h) \neq 0$, then $(\Delta_h[f])(x) = \frac{16 \cdot \cos(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{-h}{2}) \cdot \cos(\frac{-h}{2}) \cdot \sin(\frac{2 \cdot x + h}{2})}{(\cos(2 \cdot x + h) - \cos h)^2}$.

(71) If for every x holds $f(x) = \frac{1}{(\sin x)^2}$ and $\sin x \neq 0$ and $\sin(x-h) \neq 0$, then

$(\nabla_h[f])(x) = \frac{16\cdot\cos(\frac{2\cdot x-h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{-h}{2})\cdot\sin(\frac{2\cdot x-h}{2})}{(\cos(2\cdot x-h)-\cos h)^{\mathbf{2}}}$.

(72) If for every x holds $f(x) = \frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin(x+\frac{h}{2}) \neq 0$ and $\sin(x-\frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{16\cdot\cos x\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{-h}{2})\cdot\sin x}{(\cos(2\cdot x)-\cos h)^{\mathbf{2}}}$.

(73) Suppose for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $x_0 \neq x_1$ and $\cos x_0 \neq 0$ and $\cos x_1 \neq 0$. Then $\Delta[f](x_0,x_1) = \frac{\frac{(-16)\cdot\sin(\frac{x_1+x_0}{2})\cdot\sin(\frac{x_1-x_0}{2})\cdot\cos(\frac{x_1+x_0}{2})\cdot\cos(\frac{x_1-x_0}{2})}{(\cos(x_0+x_1)+\cos(x_0-x_1))^{\mathbf{2}}}}{x_0-x_1}$.

(74) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and $\cos(x+h) \neq 0$, then $(\Delta_h[f])(x) = \frac{(-16)\cdot\sin(\frac{2\cdot x+h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{2\cdot x+h}{2})\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x+h)+\cos h)^{\mathbf{2}}}$.

(75) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and $\cos(x-h) \neq 0$, then $(\nabla_h[f])(x) = \frac{(-16)\cdot\sin(\frac{2\cdot x-h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{2\cdot x-h}{2})\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x-h)+\cos h)^{\mathbf{2}}}$.

(76) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos(x+\frac{h}{2}) \neq 0$ and $\cos(x-\frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{(-16)\cdot\sin x\cdot\sin(\frac{-h}{2})\cdot\cos x\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x)+\cos h)^{\mathbf{2}}}$.

(77) Suppose $x_0 \in \operatorname{dom}$ (the function tan) and $x_1 \in \operatorname{dom}$ (the function tan). Then Δ[(the function tan) (the function sin)]$(x_0,x_1) = \frac{(\frac{1}{\cos x_0}-\cos x_0-\frac{1}{\cos x_1})+\cos x_1}{x_0-x_1}$.

(78) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x \in \operatorname{dom}$ (the function tan), and

(iii) $x+h \in \operatorname{dom}$ (the function tan).

Then $(\Delta_h[f])(x) = (\frac{1}{\cos(x+h)} - \cos(x+h) - \frac{1}{\cos x}) + \cos x$.

(79) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x \in \operatorname{dom}$ (the function tan), and

(iii) $x-h \in \operatorname{dom}$ (the function tan).

Then $(\nabla_h[f])(x) = (\frac{1}{\cos x} - \cos x - \frac{1}{\cos(x-h)}) + \cos(x-h)$.

(80) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x+\frac{h}{2} \in \operatorname{dom}$ (the function tan), and

(iii) $x-\frac{h}{2} \in \operatorname{dom}$ (the function tan).

Then $(\delta_h[f])(x) = (\frac{1}{\cos(x+\frac{h}{2})} - \cos(x+\frac{h}{2}) - \frac{1}{\cos(x-\frac{h}{2})}) + \cos(x-\frac{h}{2})$.

(81) Suppose for every x holds $f(x) =$ ((the function tan) (the function cos))(x) and $x_0 \in \operatorname{dom}$ (the function tan) and $x_1 \in \operatorname{dom}$ (the function tan). Then $\Delta[f](x_0,x_1) = \frac{\sin x_0-\sin x_1}{x_0-x_1}$.

(82) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),

(ii) $x \in \operatorname{dom}$ (the function tan), and

(iii) $x + h \in \mathrm{dom}$ (the function tan).
Then $(\Delta_h[f])(x) = \sin(x + h) - \sin x$.

(83) Suppose that
(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function tan), and
(iii) $x - h \in \mathrm{dom}$ (the function tan).
Then $(\nabla_h[f])(x) = \sin x - \sin(x - h)$.

(84) Suppose that
(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),
(ii) $x + \frac{h}{2} \in \mathrm{dom}$ (the function tan), and
(iii) $x - \frac{h}{2} \in \mathrm{dom}$ (the function tan).
Then $(\delta_h[f])(x) = \sin(x + \frac{h}{2}) - \sin(x - \frac{h}{2})$.

(85) Suppose for every x holds $f(x) =$ ((the function cot) (the function cos))(x) and $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then $\Delta[f](x_0, x_1) = \frac{(\frac{1}{\sin x_0} - \sin x_0 - \frac{1}{\sin x_1}) + \sin x_1}{x_0 - x_1}$.

(86) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x + h \in \mathrm{dom}$ (the function cot).
Then $(\Delta_h[f])(x) = (\frac{1}{\sin(x+h)} - \sin(x + h) - \frac{1}{\sin x}) + \sin x$.

(87) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x - h \in \mathrm{dom}$ (the function cot).
Then $(\nabla_h[f])(x) = (\frac{1}{\sin x} - \sin x - \frac{1}{\sin(x-h)}) + \sin(x - h)$.

(88) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x + \frac{h}{2} \in \mathrm{dom}$ (the function cot), and
(iii) $x - \frac{h}{2} \in \mathrm{dom}$ (the function cot).
Then $(\delta_h[f])(x) = (\frac{1}{\sin(x+\frac{h}{2})} - \sin(x + \frac{h}{2}) - \frac{1}{\sin(x-\frac{h}{2})}) + \sin(x - \frac{h}{2})$.

(89) Suppose for every x holds $f(x) =$ ((the function cot) (the function sin))(x) and $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then $\Delta[f](x_0, x_1) = \frac{\cos x_0 - \cos x_1}{x_0 - x_1}$.

(90) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function sin))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x + h \in \mathrm{dom}$ (the function cot).
Then $(\Delta_h[f])(x) = \cos(x + h) - \cos x$.

(91) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function sin))(x),

(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x - h \in \mathrm{dom}$ (the function cot).
Then $(\nabla_h[f])(x) = \cos x - \cos(x - h)$.

(92) Suppose that
(i) for every x holds $f(x) = $ ((the function cot) (the function sin))(x),
(ii) $x + \frac{h}{2} \in \mathrm{dom}$ (the function cot), and
(iii) $x - \frac{h}{2} \in \mathrm{dom}$ (the function cot).
Then $(\delta_h[f])(x) = \cos(x + \frac{h}{2}) - \cos(x - \frac{h}{2})$.

(93) Suppose for every x holds $f(x) = $ ((the function tan) (the function tan))(x) and $x_0 \in \mathrm{dom}$ (the function tan) and $x_1 \in \mathrm{dom}$ (the function tan). Then $\Delta[f](x_0, x_1) = \frac{(\cos x_1)^{\mathbf{2}} - (\cos x_0)^{\mathbf{2}}}{(\cos x_0 \cdot \cos x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(94) Suppose that
(i) for every x holds $f(x) = $ ((the function tan) (the function tan))(x),
(ii) $x \in \mathrm{dom}$ (the function tan), and
(iii) $x + h \in \mathrm{dom}$ (the function tan).
Then $(\Delta_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(2 \cdot (x+h)) - \cos(2 \cdot x))}{(\cos(x+h) \cdot \cos x)^{\mathbf{2}}}$.

(95) Suppose that
(i) for every x holds $f(x) = $ ((the function tan) (the function tan))(x),
(ii) $x \in \mathrm{dom}$ (the function tan), and
(iii) $x - h \in \mathrm{dom}$ (the function tan).
Then $(\nabla_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (h - x)))}{(\cos x \cdot \cos(x - h))^{\mathbf{2}}}$.

(96) Suppose that
(i) for every x holds $f(x) = $ ((the function tan) (the function tan))(x),
(ii) $x + \frac{h}{2} \in \mathrm{dom}$ (the function tan), and
(iii) $x - \frac{h}{2} \in \mathrm{dom}$ (the function tan).
Then $(\delta_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(h + 2 \cdot x) - \cos(h - 2 \cdot x))}{(\cos(x + \frac{h}{2}) \cdot \cos(x - \frac{h}{2}))^{\mathbf{2}}}$.

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