

# A Model of Mizar Concepts – Unification

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**Summary.** The aim of this paper is to develop a formal theory of Mizar linguistic concepts following the ideas from [6] and [7]. The theory presented is an abstraction from the existing implementation of the Mizar system and is devoted to the formalization of Mizar expressions. The concepts formalized here are: standardized constructor signature, arity-rich signatures, and the unification of Mizar expressions.

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The notation and terminology used in this paper are introduced in the following articles: [20], [21], [12], [22], [10], [14], [13], [17], [18], [15], [1], [8], [11], [2], [3], [4], [19], [16], [5], [9], and [7]. For simplicity the abbreviation  $\mathfrak{M} = \text{MaxConstrSign}$  is introduced.

## 1. PRELIMINARY

In this paper  $i, j$  denote natural numbers.

Next we state two propositions:

- (1) For every pair set  $x$  holds  $x = \langle x_1, x_2 \rangle$ .
- (2) For every infinite set  $X$  there exist sets  $x_1, x_2$  such that  $x_1, x_2 \in X$  and  $x_1 \neq x_2$ .

In this article we present several logical schemes. The scheme *MinimalElement* deals with a finite non empty set  $\mathcal{A}$  and a binary predicate  $\mathcal{P}$ , and states that:

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There exists a set  $x$  such that  $x \in \mathcal{A}$  and for every set  $y$  such that  $y \in \mathcal{A}$  holds not  $\mathcal{P}[y, x]$

provided the parameters have the following properties:

- For all sets  $x, y$  such that  $x, y \in \mathcal{A}$  and  $\mathcal{P}[x, y]$  holds not  $\mathcal{P}[y, x]$ , and
- For all sets  $x, y, z$  such that  $x, y, z \in \mathcal{A}$  and  $\mathcal{P}[x, y]$  and  $\mathcal{P}[y, z]$  holds  $\mathcal{P}[x, z]$ .

The scheme *FiniteC* deals with a finite set  $\mathcal{A}$  and a unary predicate  $\mathcal{P}$ , and states that:

$\mathcal{P}[\mathcal{A}]$

provided the following condition is satisfied:

- For every subset  $A$  of  $\mathcal{A}$  such that for every set  $B$  such that  $B \subset A$  holds  $\mathcal{P}[B]$  holds  $\mathcal{P}[A]$ .

The scheme *Numeration* deals with a finite set  $\mathcal{A}$  and a binary predicate  $\mathcal{P}$ , and states that:

There exists an one-to-one finite sequence  $s$  such that  $\text{rng } s = \mathcal{A}$  and for all  $i, j$  such that  $i, j \in \text{dom } s$  and  $\mathcal{P}[s(i), s(j)]$  holds  $i < j$

provided the parameters satisfy the following conditions:

- For all sets  $x, y$  such that  $x, y \in \mathcal{A}$  and  $\mathcal{P}[x, y]$  holds not  $\mathcal{P}[y, x]$ , and
- For all sets  $x, y, z$  such that  $x, y, z \in \mathcal{A}$  and  $\mathcal{P}[x, y]$  and  $\mathcal{P}[y, z]$  holds  $\mathcal{P}[x, z]$ .

One can prove the following two propositions:

- (3) For every variable  $x$  holds  $\text{varcl vars}(x) = \text{vars}(x)$ .
- (4) Let  $\mathfrak{C}$  be an initialized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Then  $e$  is compound if and only if it is not true that there exists an element  $x$  of  $\text{Vars}$  such that  $e = x_{\mathfrak{C}}$ .

## 2. STANDARDIZED CONSTRUCTOR SIGNATURE

Let us note that there exists a quasi-locus sequence which is empty.

Let  $\mathfrak{C}$  be a constructor signature. We say that  $\mathfrak{C}$  is standardized if and only if the condition (Def. 1) is satisfied.

- (Def. 1) Let  $o$  be an operation symbol of  $\mathfrak{C}$ . Suppose  $o$  is constructor. Then  $o \in \text{Constructors}$  and  $o_1 = \text{the result sort of } o$  and  $\text{Card}((o_2)_1) = \text{len Arity}(o)$ .

The following proposition is true

- (5) Let  $\mathfrak{C}$  be a constructor signature. Suppose  $\mathfrak{C}$  is standardized. Let  $o$  be an operation symbol of  $\mathfrak{C}$ . Then  $o$  is constructor if and only if  $o \in \text{Constructors}$ .

Let us note that  $\mathfrak{M}$  is standardized.

Let us observe that there exists a constructor signature which is initialized, standardized, and strict.

Let  $\mathfrak{C}$  be an initialized standardized constructor signature and let  $c$  be a constructor operation symbol of  $\mathfrak{C}$ . The loci of  $c$  yielding a quasi-locus sequence is defined by:

(Def. 2) The loci of  $c = (c_2)_1$ .

Let  $\mathfrak{C}$  be a constructor signature. One can verify that there exists a subsignature of  $\mathfrak{C}$  which is constructor.

Let  $\mathfrak{C}$  be an initialized constructor signature. Note that there exists a constructor subsignature of  $\mathfrak{C}$  which is initialized.

Let  $\mathfrak{C}$  be a standardized constructor signature. One can verify that every constructor subsignature of  $\mathfrak{C}$  is standardized.

One can prove the following two propositions:

- (6) Let  $S_1, S_2$  be standardized constructor signatures. Suppose the operation symbols of  $S_1 =$  the operation symbols of  $S_2$ . Then the many sorted signature of  $S_1 =$  the many sorted signature of  $S_2$ .
- (7) For every constructor signature  $\mathfrak{C}$  holds  $\mathfrak{C}$  is standardized iff  $\mathfrak{C}$  is a subsignature of  $\mathfrak{M}$ .

Let  $\mathfrak{C}$  be an initialized constructor signature. Observe that there exists a quasi-term of  $\mathfrak{C}$  which is non compound.

Let us mention that every element of Vars is pair.

The following propositions are true:

- (8) For every element  $x$  of Vars such that  $\text{vars}(x)$  is natural holds  $\text{vars}(x) = 0$ .
- (9) Vars misses Constructors.
- (10) For every element  $x$  of Vars holds  $x \neq *$  and  $x \neq \mathbf{non}$ .
- (11) For every standardized constructor signature  $\mathfrak{C}$  holds Vars misses the operation symbols of  $\mathfrak{C}$ .
- (12) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Then
  - (i) there exists an element  $x$  of Vars such that  $e = x_{\mathfrak{C}}$  and  $e(\emptyset) = \langle x, \mathbf{term} \rangle$ , or
  - (ii) there exists an operation symbol  $o$  of  $\mathfrak{C}$  such that  $e(\emptyset) = \langle o, \text{the carrier of } \mathfrak{C} \rangle$  but  $o \in \text{Constructors}$  or  $o = *$  or  $o = \mathbf{non}$ .

Let  $\mathfrak{C}$  be an initialized standardized constructor signature and let  $e$  be an expression of  $\mathfrak{C}$ . Note that  $e(\emptyset)$  is pair.

The following propositions are true:

- (13) Let  $\mathfrak{C}$  be an initialized constructor signature,  $e$  be an expression of  $\mathfrak{C}$ , and  $o$  be an operation symbol of  $\mathfrak{C}$ . Suppose  $e(\emptyset) = \langle o, \text{the carrier of } \mathfrak{C} \rangle$ . Then  $e$  is an expression of  $\mathfrak{C}$  from the result sort of  $o$ .

- (14) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Then
- (i) if  $e(\emptyset)_1 = *$ , then  $e$  is an expression of  $\mathfrak{C}$  from  $\mathbf{type}_{\mathfrak{C}}$ , and
  - (ii) if  $e(\emptyset)_1 = \mathbf{non}$ , then  $e$  is an expression of  $\mathfrak{C}$  from  $\mathbf{adj}_{\mathfrak{C}}$ .
- (15) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Then
- (i)  $e(\emptyset)_1 \in \mathbf{Vars}$  and  $e(\emptyset)_2 = \mathbf{term}$  and  $e$  is a quasi-term of  $\mathfrak{C}$ , or
  - (ii)  $e(\emptyset)_2 =$  the carrier of  $\mathfrak{C}$  but  $e(\emptyset)_1 \in \mathbf{Constructors}$  and  $e(\emptyset)_1 \in$  the operation symbols of  $\mathfrak{C}$  or  $e(\emptyset)_1 = *$  or  $e(\emptyset)_1 = \mathbf{non}$ .
- (16) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . If  $e(\emptyset)_1 \in \mathbf{Constructors}$ , then  $e \in$  (the sorts of  $\mathbf{Free}_{\mathfrak{C}}(\mathbf{Vars} \mathfrak{C})((e(\emptyset)_1)_1)$ ).
- (17) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Then  $e(\emptyset)_1 \notin \mathbf{Vars}$  if and only if  $e(\emptyset)_1$  is an operation symbol of  $\mathfrak{C}$ .
- (18) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . If  $e(\emptyset)_1 \in \mathbf{Vars}$ , then there exists an element  $x$  of  $\mathbf{Vars}$  such that  $x = e(\emptyset)_1$  and  $e = x_{\mathfrak{C}}$ .
- (19) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Suppose  $e(\emptyset)_1 = *$ . Then there exists an expression  $\alpha$  of  $\mathfrak{C}$  from  $\mathbf{adj}_{\mathfrak{C}}$  and there exists an expression  $q$  of  $\mathfrak{C}$  from  $\mathbf{type}_{\mathfrak{C}}$  such that  $e = \langle *, 3 \rangle\text{-tree}(\alpha, q)$ .
- (20) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . If  $e(\emptyset)_1 = \mathbf{non}$ , then there exists an expression  $\alpha$  of  $\mathfrak{C}$  from  $\mathbf{adj}_{\mathfrak{C}}$  such that  $e = \langle \mathbf{non}, 3 \rangle\text{-tree}(\alpha)$ .
- (21) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $e$  be an expression of  $\mathfrak{C}$ . Suppose  $e(\emptyset)_1 \in \mathbf{Constructors}$ . Then there exists an operation symbol  $o$  of  $\mathfrak{C}$  such that  $o = e(\emptyset)_1$  and the result sort of  $o = o_1$  and  $e$  is an expression of  $\mathfrak{C}$  from the result sort of  $o$ .
- (22) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\tau$  be a quasi-term of  $\mathfrak{C}$ . Then  $\tau$  is compound if and only if  $\tau(\emptyset)_1 \in \mathbf{Constructors}$  and  $(\tau(\emptyset)_1)_1 = \mathbf{term}$ .
- (23) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\tau$  be an expression of  $\mathfrak{C}$ . Then  $\tau$  is a non compound quasi-term of  $\mathfrak{C}$  if and only if  $\tau(\emptyset)_1 \in \mathbf{Vars}$ .
- (24) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\tau$  be an expression of  $\mathfrak{C}$ . Then  $\tau$  is a quasi-term of  $\mathfrak{C}$  if and only if  $\tau(\emptyset)_1 \in \mathbf{Constructors}$  and  $(\tau(\emptyset)_1)_1 = \mathbf{term}$  or  $\tau(\emptyset)_1 \in \mathbf{Vars}$ .
- (25) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\alpha$  be an expression of  $\mathfrak{C}$ . Then  $\alpha$  is a positive quasi-adjective of  $\mathfrak{C}$  if and only if

- $\alpha(\emptyset)_1 \in \text{Constructors}$  and  $(\alpha(\emptyset)_1)_1 = \mathbf{adj}$ .
- (26) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\alpha$  be a quasi-adjective of  $\mathfrak{C}$ . Then  $\alpha$  is negative if and only if  $\alpha(\emptyset)_1 = \mathbf{non}$ .
- (27) Let  $\mathfrak{C}$  be an initialized standardized constructor signature and  $\tau$  be an expression of  $\mathfrak{C}$ . Then  $\tau$  is a pure expression of  $\mathfrak{C}$  from  $\mathbf{type}_{\mathfrak{C}}$  if and only if  $\tau(\emptyset)_1 \in \text{Constructors}$  and  $(\tau(\emptyset)_1)_1 = \mathbf{type}$ .

### 3. EXPRESSIONS

In the sequel  $i$  is a natural number,  $x$  is a variable, and  $\ell$  is a quasi-locus sequence.

An expression is an expression of  $\mathfrak{M}$ . A valuation is a valuation of  $\mathfrak{M}$ . A quasi-adjective is a quasi-adjective of  $\mathfrak{M}$ . The subset  $\text{QuasiAdjs}$  of  $\text{Free}_{\mathfrak{M}}(\text{Vars } \mathfrak{M})$  is defined as follows:

(Def. 3)  $\text{QuasiAdjs} = \text{QuasiAdjs } \mathfrak{M}$ .

A quasi-term is a quasi-term of  $\mathfrak{M}$ . The subset  $\text{QuasiTerms}$  of  $\text{Free}_{\mathfrak{M}}(\text{Vars } \mathfrak{M})$  is defined as follows:

(Def. 4)  $\text{QuasiTerms} = \text{QuasiTerms } \mathfrak{M}$ .

A quasi-type is a quasi-type of  $\mathfrak{M}$ . The functor  $\text{QuasiTypes}$  is defined as follows:

(Def. 5)  $\text{QuasiTypes} = \text{QuasiTypes } \mathfrak{M}$ .

One can verify the following observations:

- \*  $\text{QuasiAdjs}$  is non empty,
- \*  $\text{QuasiTerms}$  is non empty, and
- \*  $\text{QuasiTypes}$  is non empty.

$\text{Modes}$  is a non empty subset of  $\text{Constructors}$ . Then  $\text{Attrs}$  is a non empty subset of  $\text{Constructors}$ . Then  $\text{Funcs}$  is a non empty subset of  $\text{Constructors}$ .

In the sequel  $\mathfrak{C}$  denotes an initialized constructor signature.

The element set-constr of  $\text{Modes}$  is defined by:

(Def. 6)  $\text{set-constr} = \langle \mathbf{type}, \langle \emptyset, 0 \rangle \rangle$ .

One can prove the following propositions:

- (28) The kind of  $\text{set-constr} = \mathbf{type}$  and the loci of  $\text{set-constr} = \emptyset$  and the index of  $\text{set-constr} = 0$ .
- (29)  $\text{Constructors} = \{\mathbf{type}, \mathbf{adj}, \mathbf{term}\} \times (\text{QuasiLoci} \times \mathbb{N})$ .
- (30)  $\langle \text{rng } \ell, i \rangle \in \text{Vars}$  and  $\ell \hat{\ } \langle \langle \text{rng } \ell, i \rangle \rangle$  is a quasi-locus sequence.
- (31) There exists  $\ell$  such that  $\text{len } \ell = i$ .
- (32) For every finite subset  $X$  of  $\text{Vars}$  there exists  $\ell$  such that  $\text{rng } \ell = \text{varcl } X$ .
- (33) Let  $X, o$  be sets and  $p$  be a decorated tree yielding finite sequence. Given  $\mathfrak{C}$  such that  $X = \bigcup (\text{the sorts of } \text{Free}_{\mathfrak{C}}(\text{Vars } \mathfrak{C}))$ . If  $o\text{-tree}(p) \in X$ , then  $p$  is a finite sequence of elements of  $X$ .

Let us consider  $\mathfrak{C}$  and let  $e$  be an expression of  $\mathfrak{C}$ . An expression of  $\mathfrak{C}$  is called a subexpression of  $e$  if:

(Def. 7)  $It \in \text{Subtrees}(e)$ .

The functor  $\text{constrs } e$  is defined by:

(Def. 8)  $\text{constrs } e = \pi_1(\text{rng } e) \cap \{o : o \text{ ranges over constructor operation symbols of } \mathfrak{C}\}$ .

The functor  $\text{main-constr } e$  is defined by:

(Def. 9)  $\text{main-constr } e = \begin{cases} e(\emptyset)_1, & \text{if } e \text{ is compound,} \\ \emptyset, & \text{otherwise.} \end{cases}$

The functor  $\text{args } e$  yields a finite sequence of elements of  $\text{Free}_{\mathfrak{C}}(\text{Vars } \mathfrak{C})$  and is defined by:

(Def. 10)  $e = e(\emptyset)\text{-tree}(\text{args } e)$ .

Next we state three propositions:

(34) For every  $\mathfrak{C}$  holds every expression  $e$  of  $\mathfrak{C}$  is a subexpression of  $e$ .

(35)  $\text{main-constr}(x_{\mathfrak{C}}) = \emptyset$ .

(36) Let  $c$  be a constructor operation symbol of  $\mathfrak{C}$  and  $p$  be a finite sequence of elements of  $\text{QuasiTerms } \mathfrak{C}$ . If  $\text{len } p = \text{len Arity}(c)$ , then  $\text{main-constr}(c^{\neg}(p)) = c$ .

Let us consider  $\mathfrak{C}$  and let  $e$  be an expression of  $\mathfrak{C}$ . We say that  $e$  is constructor if and only if:

(Def. 11)  $e$  is compound and  $\text{main-constr } e$  is a constructor operation symbol of  $\mathfrak{C}$ .

Let us consider  $\mathfrak{C}$ . Observe that every expression of  $\mathfrak{C}$  which is constructor is also compound.

Let us consider  $\mathfrak{C}$ . Observe that there exists an expression of  $\mathfrak{C}$  which is constructor.

Let us consider  $\mathfrak{C}$  and let  $e$  be a constructor expression of  $\mathfrak{C}$ . One can verify that there exists a subexpression of  $e$  which is constructor.

Let  $S$  be a non void signature, let  $X$  be a non empty yielding many sorted set indexed by  $S$ , and let  $\tau$  be an element of  $\text{Free}_S(X)$ . Observe that  $\text{rng } \tau$  is relation-like.

One can prove the following proposition

(37) For every constructor expression  $e$  of  $\mathfrak{C}$  holds  $\text{main-constr } e \in \text{constrs } e$ .

#### 4. ARITY

For simplicity, we follow the rules:  $\alpha$  is a quasi-adjective,  $\tau, \tau_1, \tau_2$  are quasi-terms,  $\vartheta$  is a quasi-type, and  $c$  is an element of Constructors.

Let  $\mathfrak{C}$  be a non void signature. We say that  $\mathfrak{C}$  is arity-rich if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let  $n$  be a natural number and  $s$  be a sort symbol of  $\mathfrak{C}$ . Then  $\{o; o \text{ ranges over operation symbols of } \mathfrak{C}; \text{ the result sort of } o = s \wedge \text{len Arity}(o) = n\}$  is infinite.

Let  $o$  be an operation symbol of  $\mathfrak{C}$ . We say that  $o$  is nullary if and only if:

(Def. 13)  $\text{Arity}(o) = \emptyset$ .

We say that  $o$  is unary if and only if:

(Def. 14)  $\text{len Arity}(o) = 1$ .

We say that  $o$  is binary if and only if:

(Def. 15)  $\text{len Arity}(o) = 2$ .

The following proposition is true

- (38) Let  $\mathfrak{C}$  be a non void signature and  $o$  be an operation symbol of  $\mathfrak{C}$ . Then
- (i) if  $o$  is nullary, then  $o$  is not unary,
  - (ii) if  $o$  is nullary, then  $o$  is not binary, and
  - (iii) if  $o$  is unary, then  $o$  is not binary.

Let  $\mathfrak{C}$  be a constructor signature. Observe that  $\mathbf{non}_{\mathfrak{C}}$  is unary and  $*_{\mathfrak{C}}$  is binary.

Let  $\mathfrak{C}$  be a constructor signature. Note that every operation symbol of  $\mathfrak{C}$  which is nullary is also constructor.

The following proposition is true

- (39) Let  $\mathfrak{C}$  be a constructor signature. Then  $\mathfrak{C}$  is initialized if and only if there exists an operation symbol  $m$  of  $\mathbf{type}_{\mathfrak{C}}$  and there exists an operation symbol  $\alpha$  of  $\mathbf{adj}_{\mathfrak{C}}$  such that  $m$  is nullary and  $\alpha$  is nullary.

Let  $\mathfrak{C}$  be an initialized constructor signature. One can verify that there exists an operation symbol of  $\mathbf{type}_{\mathfrak{C}}$  which is nullary and constructor and there exists an operation symbol of  $\mathbf{adj}_{\mathfrak{C}}$  which is nullary and constructor.

Let  $\mathfrak{C}$  be an initialized constructor signature. Observe that there exists an operation symbol of  $\mathfrak{C}$  which is nullary and constructor.

One can check that every non void signature which is arity-rich has also an operation for each sort and every constructor signature which is arity-rich is also initialized.

One can check that  $\mathfrak{M}$  is arity-rich.

Let us mention that there exists a constructor signature which is arity-rich and initialized.

Let  $\mathfrak{C}$  be an arity-rich constructor signature and let  $s$  be a sort symbol of  $\mathfrak{C}$ . One can verify the following observations:

- \* there exists an operation symbol of  $s$  which is nullary and constructor,
- \* there exists an operation symbol of  $s$  which is unary and constructor, and
- \* there exists an operation symbol of  $s$  which is binary and constructor.

Let  $\mathfrak{C}$  be an arity-rich constructor signature. One can check that there exists an operation symbol of  $\mathfrak{C}$  which is unary and constructor and there exists an operation symbol of  $\mathfrak{C}$  which is binary and constructor.

The following proposition is true

- (40) Let  $o$  be a nullary operation symbol of  $\mathfrak{C}$ . Then  $\langle o, \text{the carrier of } \mathfrak{C}\text{-tree}(\emptyset) \rangle$  is an expression of  $\mathfrak{C}$  from the result sort of  $o$ .

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $m$  be a nullary constructor operation symbol of  $\mathbf{type}_{\mathfrak{C}}$ . Then  $m_t$  is a pure expression of  $\mathfrak{C}$  from  $\mathbf{type}_{\mathfrak{C}}$ .

Let  $c$  be an element of Constructors. The functor  ${}^{\textcircled{a}}c$  yielding a constructor operation symbol of  $\mathfrak{M}$  is defined by:

- (Def. 16)  ${}^{\textcircled{a}}c = c$ .

Let  $m$  be an element of Modes. Then  ${}^{\textcircled{a}}m$  is a constructor operation symbol of  $\mathbf{type}_{\mathfrak{M}}$ .

Let us note that  ${}^{\textcircled{a}}\text{set-constr}$  is nullary.

We now state the proposition

- (41)  $\text{Arity}({}^{\textcircled{a}}\text{set-constr}) = \emptyset$ .

The quasi-type set-type is defined by:

- (Def. 17)  $\text{set-type} = \emptyset_{\text{QuasiAdjs } \mathfrak{M}} * ({}^{\textcircled{a}}\text{set-constr})_t$ .

The following proposition is true

- (42)  $\text{adjs set-type} = \emptyset$  and the base of  $\text{set-type} = ({}^{\textcircled{a}}\text{set-constr})_t$ .

Let  $\ell$  be a finite sequence of elements of Vars. The functor  $\text{args } \ell$  yields a finite sequence of elements of QuasiTerms  $\mathfrak{M}$  and is defined as follows:

- (Def. 18)  $\text{len args } \ell = \text{len } \ell$  and for every  $i$  such that  $i \in \text{dom } \ell$  holds  $(\text{args } \ell)(i) = (\ell_i)_{\mathfrak{M}}$ .

Let us consider  $c$ . The base expression of  $c$  yields an expression and is defined as follows:

- (Def. 19) The base expression of  $c = ({}^{\textcircled{a}}c)^{\neg}(\text{args}(\text{the loci of } c))$ .

Next we state several propositions:

- (43) For every operation symbol  $o$  of  $\mathfrak{M}$  holds  $o$  is constructor iff  $o \in \text{Constructors}$ .
- (44) For every nullary operation symbol  $m$  of  $\mathfrak{M}$  holds  $\text{main-constr}(m_t) = m$ .
- (45) For every unary constructor operation symbol  $m$  of  $\mathfrak{M}$  and for every  $\tau$  holds  $\text{main-constr}(m(\tau)) = m$ .
- (46) For every  $\alpha$  holds  $\text{main-constr}(\mathbf{non}_{\mathfrak{M}}(\alpha)) = \mathbf{non}$ .
- (47) For every binary constructor operation symbol  $m$  of  $\mathfrak{M}$  and for all  $\tau_1, \tau_2$  holds  $\text{main-constr}(m(\tau_1, \tau_2)) = m$ .
- (48) For every expression  $q$  of  $\mathfrak{M}$  from  $\mathbf{type}_{\mathfrak{M}}$  and for every  $\alpha$  holds  $\text{main-constr}(*_{\mathfrak{M}}(\alpha, q)) = *$ .

Let  $\vartheta$  be a quasi-type. The functor  $\text{constrs } \vartheta$  is defined by:

(Def. 20)  $\text{constrs } \vartheta = \text{constrs}(\text{the base of } \vartheta) \cup \bigcup \{\text{constrs } \alpha : \alpha \in \text{adjs } \vartheta\}$ .

The following two propositions are true:

(49) For every pure expression  $q$  of  $\mathfrak{M}$  from  $\mathbf{type}_{\mathfrak{M}}$  and for every finite subset  $A$  of  $\text{QuasiAdjs } \mathfrak{M}$  holds  $\text{constrs}(A * q) = \text{constrs } q \cup \bigcup \{\text{constrs } \alpha : \alpha \in A\}$ .

(50)  $\text{constrs}(\alpha * \vartheta) = \text{constrs } \alpha \cup \text{constrs } \vartheta$ .

## 5. UNIFICATION

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\tau, p$  be expressions of  $\mathfrak{C}$ . We say that  $\tau$  matches  $p$  if and only if:

(Def. 21) There exists a valuation  $f$  of  $\mathfrak{C}$  such that  $\tau = p[f]$ .

Let us note that the predicate  $\tau$  matches  $p$  is reflexive.

The following proposition is true

(51) For all expressions  $\tau_1, \tau_2, \tau_3$  of  $\mathfrak{C}$  such that  $\tau_1$  matches  $\tau_2$  and  $\tau_2$  matches  $\tau_3$  holds  $\tau_1$  matches  $\tau_3$ .

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $A, B$  be subsets of  $\text{QuasiAdjs } \mathfrak{C}$ . We say that  $A$  matches  $B$  if and only if:

(Def. 22) There exists a valuation  $f$  of  $\mathfrak{C}$  such that  $B[f] \subseteq A$ .

Let us note that the predicate  $A$  matches  $B$  is reflexive.

The following proposition is true

(52) For all subsets  $A_1, A_2, A_3$  of  $\text{QuasiAdjs } \mathfrak{C}$  such that  $A_1$  matches  $A_2$  and  $A_2$  matches  $A_3$  holds  $A_1$  matches  $A_3$ .

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\vartheta, P$  be quasi-types of  $\mathfrak{C}$ . We say that  $\vartheta$  matches  $P$  if and only if:

(Def. 23) There exists a valuation  $f$  of  $\mathfrak{C}$  such that  $(\text{adjs } P)[f] \subseteq \text{adjs } \vartheta$  and  $(\text{the base of } P)[f] = \text{the base of } \vartheta$ .

Let us note that the predicate  $\vartheta$  matches  $P$  is reflexive.

One can prove the following proposition

(53) For all quasi-types  $\vartheta_1, \vartheta_2, \vartheta_3$  of  $\mathfrak{C}$  such that  $\vartheta_1$  matches  $\vartheta_2$  and  $\vartheta_2$  matches  $\vartheta_3$  holds  $\vartheta_1$  matches  $\vartheta_3$ .

Let  $\mathfrak{C}$  be an initialized constructor signature, let  $\tau_1, \tau_2$  be expressions of  $\mathfrak{C}$ , and let  $f$  be a valuation of  $\mathfrak{C}$ . We say that  $f$  unifies  $\tau_1$  with  $\tau_2$  if and only if:

(Def. 24)  $\tau_1[f] = \tau_2[f]$ .

The following proposition is true

(54) Let  $\tau_1, \tau_2$  be expressions of  $\mathfrak{C}$  and  $f$  be a valuation of  $\mathfrak{C}$ . If  $f$  unifies  $\tau_1$  with  $\tau_2$ , then  $f$  unifies  $\tau_2$  with  $\tau_1$ .

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\tau_1, \tau_2$  be expressions of  $\mathfrak{C}$ . We say that  $\tau_1$  and  $\tau_2$  are unifiable if and only if:

(Def. 25) There exists a valuation  $f$  of  $\mathfrak{C}$  such that  $f$  unifies  $\tau_1$  with  $\tau_2$ .

Let us notice that the predicate  $\tau_1$  and  $\tau_2$  are unifiable is reflexive and symmetric.

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\tau_1, \tau_2$  be expressions of  $\mathfrak{C}$ . We say that  $\tau_1$  and  $\tau_2$  are weakly-unifiable if and only if:

(Def. 26) There exists an irrelevant one-to-one valuation  $g$  of  $\mathfrak{C}$  such that  $\text{Var } \tau_2 \subseteq \text{dom } g$  and  $\tau_1$  and  $\tau_2[g]$  are unifiable.

Let us note that the predicate  $\tau_1$  and  $\tau_2$  are weakly-unifiable is reflexive.

We now state the proposition

(55) For all expressions  $\tau_1, \tau_2$  of  $\mathfrak{C}$  such that  $\tau_1$  and  $\tau_2$  are unifiable holds  $\tau_1$  and  $\tau_2$  are weakly-unifiable.

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\tau, \tau_1, \tau_2$  be expressions of  $\mathfrak{C}$ . We say that  $\tau$  is a unification of  $\tau_1$  and  $\tau_2$  if and only if:

(Def. 27) There exists a valuation  $f$  of  $\mathfrak{C}$  such that  $f$  unifies  $\tau_1$  with  $\tau_2$  and  $\tau = \tau_1[f]$ .

We now state two propositions:

(56) For all expressions  $\tau_1, \tau_2, \tau$  of  $\mathfrak{C}$  such that  $\tau$  is a unification of  $\tau_1$  and  $\tau_2$  holds  $\tau$  is a unification of  $\tau_2$  and  $\tau_1$ .

(57) For all expressions  $\tau_1, \tau_2, \tau$  of  $\mathfrak{C}$  such that  $\tau$  is a unification of  $\tau_1$  and  $\tau_2$  holds  $\tau$  matches  $\tau_1$  and  $\tau$  matches  $\tau_2$ .

Let  $\mathfrak{C}$  be an initialized constructor signature and let  $\tau, \tau_1, \tau_2$  be expressions of  $\mathfrak{C}$ . We say that  $\tau$  is a general-unification of  $\tau_1$  and  $\tau_2$  if and only if the conditions (Def. 28) are satisfied.

(Def. 28)(i)  $\tau$  is a unification of  $\tau_1$  and  $\tau_2$ , and  
(ii) for every expression  $u$  of  $\mathfrak{C}$  such that  $u$  is a unification of  $\tau_1$  and  $\tau_2$  holds  $u$  matches  $\tau$ .

## 6. TYPE DISTRIBUTION

The following three propositions are true:

(58) Let  $n$  be a natural number and  $s$  be a sort symbol of  $\mathfrak{M}$ . Then there exists a constructor operation symbol  $m$  of  $s$  such that  $\text{len Arity}(m) = n$ .

(59) Let given  $\ell, s$  be a sort symbol of  $\mathfrak{M}$ , and  $m$  be a constructor operation symbol of  $s$ . If  $\text{len Arity}(m) = \text{len } \ell$ , then  $\text{Var}(m^{\rightarrow}(\text{args } \ell)) = \text{rng } \ell$ .

(60) Let  $X$  be a finite subset of  $\text{Vars}$ . Suppose  $\text{varcl } X = X$ . Let  $s$  be a sort symbol of  $\mathfrak{M}$ . Then there exists a constructor operation symbol  $m$  of  $s$  and there exists a finite sequence  $p$  of elements of  $\text{QuasiTerms } \mathfrak{M}$  such that  $\text{len } p = \text{len Arity}(m)$  and  $\text{vars}(m^{\rightarrow}(p)) = X$ .

Let  $d$  be a partial function from  $\text{Vars}$  to  $\text{QuasiTypes}$ . We say that  $d$  is even if and only if:

(Def. 29) For all  $x, \vartheta$  such that  $x \in \text{dom } d$  and  $\vartheta = d(x)$  holds  $\text{vars}(\vartheta) = \text{vars}(x)$ .

Let  $\ell$  be a quasi-locus sequence. A partial function from Vars to QuasiTypes is said to be a type-distribution for  $\ell$  if:

(Def. 30)  $\text{dom it} = \text{rng } \ell$  and it is even.

We now state the proposition

(61) For every empty quasi-locus sequence  $\ell$  holds  $\emptyset$  is a type-distribution for  $\ell$ .

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