

# Representation of the Fibonacci and Lucas Numbers in Terms of Floor and Ceiling

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**Summary.** In the paper we show how to express the Fibonacci numbers and Lucas numbers using the floor and ceiling operations.

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The notation and terminology used here have been introduced in the following papers: [7], [3], [8], [11], [10], [1], [4], [6], [2], [5], and [9].

## 1. PRELIMINARIES

One can prove the following propositions:

- (1) For all real numbers  $a, b$  and for every natural number  $c$  holds  $(\frac{a}{b})^c = \frac{a^c}{b^c}$ .
- (2) For every real number  $a$  and for all integer numbers  $b, c$  such that  $a \neq 0$  holds  $a^{b+c} = a^b \cdot a^c$ .
- (3) For every natural number  $n$  and for every real number  $a$  such that  $n$  is even and  $a \neq 0$  holds  $(-a)^n = a^n$ .
- (4) For every natural number  $n$  and for every real number  $a$  such that  $n$  is odd and  $a \neq 0$  holds  $(-a)^n = -a^n$ .
- (5)  $|\bar{\tau}| < 1$ .
- (6) For every natural number  $n$  and for every non empty real number  $r$  such that  $n$  is even holds  $r^n > 0$ .
- (7) For every natural number  $n$  and for every real number  $r$  such that  $n$  is odd and  $r < 0$  holds  $r^n < 0$ .

- (8) For every natural number  $n$  such that  $n \neq 0$  holds  $\bar{\tau}^n < \frac{1}{2}$ .
- (9) For all natural numbers  $n, m$  and for every real number  $r$  such that  $m$  is odd and  $n \geq m$  and  $r < 0$  and  $r > -1$  holds  $r^n \geq r^m$ .
- (10) For all natural numbers  $n, m$  such that  $m$  is odd and  $n \geq m$  holds  $\bar{\tau}^n \geq \bar{\tau}^m$ .
- (11) For all natural numbers  $n, m$  such that  $n$  is even and  $m$  is even and  $n \geq m$  holds  $\bar{\tau}^n \leq \bar{\tau}^m$ .
- (12) For all non empty natural numbers  $m, n$  such that  $m \geq n$  holds  $\text{Luc}(m) \geq \text{Luc}(n)$ .
- (13) For every non empty natural number  $n$  holds  $\tau^n > \bar{\tau}^n$ .
- (14) For every natural number  $n$  such that  $n > 1$  holds  $-\frac{1}{2} < \bar{\tau}^n$ .
- (15) For every natural number  $n$  such that  $n > 2$  holds  $\bar{\tau}^n \geq -\frac{1}{\sqrt{5}}$ .
- (16) For every natural number  $n$  such that  $n \geq 2$  holds  $\bar{\tau}^n \leq \frac{1}{\sqrt{5}}$ .
- (17) For every natural number  $n$  holds  $\frac{\bar{\tau}^n}{\sqrt{5}} + \frac{1}{2} > 0$  and  $\frac{\bar{\tau}^n}{\sqrt{5}} + \frac{1}{2} < 1$ .

## 2. FORMULAS FOR THE FIBONACCI NUMBERS

Next we state two propositions:

- (18) For every natural number  $n$  holds  $\lfloor \frac{\tau^n}{\sqrt{5}} + \frac{1}{2} \rfloor = \text{Fib}(n)$ .
- (19) For every natural number  $n$  such that  $n \neq 0$  holds  $\lceil \frac{\tau^n}{\sqrt{5}} - \frac{1}{2} \rceil = \text{Fib}(n)$ .

We now state a number of propositions:

- (20) For every natural number  $n$  such that  $n \neq 0$  holds  $\lfloor \frac{\tau^{2 \cdot n}}{\sqrt{5}} \rfloor = \text{Fib}(2 \cdot n)$ .
- (21) For every natural number  $n$  holds  $\lceil \frac{\tau^{2 \cdot n + 1}}{\sqrt{5}} \rceil = \text{Fib}(2 \cdot n + 1)$ .
- (22) For every natural number  $n$  such that  $n \geq 2$  and  $n$  is even holds  $\text{Fib}(n + 1) = \lfloor \tau \cdot \text{Fib}(n) + 1 \rfloor$ .
- (23) For every natural number  $n$  such that  $n \geq 2$  and  $n$  is odd holds  $\text{Fib}(n + 1) = \lceil \tau \cdot \text{Fib}(n) - 1 \rceil$ .
- (24) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Fib}(n + 1) = \lfloor \frac{\text{Fib}(n) + \sqrt{5} \cdot \text{Fib}(n) + 1}{2} \rfloor$ .
- (25) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Fib}(n + 1) = \lceil \frac{(\text{Fib}(n) + \sqrt{5} \cdot \text{Fib}(n)) - 1}{2} \rceil$ .
- (26) For every natural number  $n$  holds  $\text{Fib}(n + 1) = \frac{\text{Fib}(n) + \sqrt{5 \cdot \text{Fib}(n)^2 + 4 \cdot (-1)^n}}{2}$ .
- (27) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Fib}(n + 1) = \lfloor \frac{\text{Fib}(n) + 1 + \sqrt{(5 \cdot \text{Fib}(n)^2 - 2 \cdot \text{Fib}(n)) + 1}}{2} \rfloor$ .
- (28) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Fib}(n) = \lfloor \frac{1}{\tau} \cdot (\text{Fib}(n + 1) + \frac{1}{2}) \rfloor$ .

- (29) For all natural numbers  $n, k$  such that  $n \geq k > 1$  or  $k = 1$  and  $n > k$  holds  $\lfloor \tau^k \cdot \text{Fib}(n) + \frac{1}{2} \rfloor = \text{Fib}(n + k)$ .

### 3. FORMULAS FOR THE LUCAS NUMBERS

Next we state a number of propositions:

- (30) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Luc}(n) = \lfloor \tau^n + \frac{1}{2} \rfloor$ .
- (31) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Luc}(n) = \lceil \tau^n - \frac{1}{2} \rceil$ .
- (32) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Luc}(2 \cdot n) = \lceil \tau^{2 \cdot n} \rceil$ .
- (33) For every natural number  $n$  such that  $n \geq 2$  holds  $\text{Luc}(2 \cdot n + 1) = \lfloor \tau^{2 \cdot n + 1} \rfloor$ .
- (34) For every natural number  $n$  such that  $n \geq 2$  and  $n$  is odd holds  $\text{Luc}(n + 1) = \lfloor \tau \cdot \text{Luc}(n) + 1 \rfloor$ .
- (35) For every natural number  $n$  such that  $n \geq 2$  and  $n$  is even holds  $\text{Luc}(n + 1) = \lceil \tau \cdot \text{Luc}(n) - 1 \rceil$ .
- (36) For every natural number  $n$  such that  $n \neq 1$  holds  $\text{Luc}(n + 1) = \frac{\text{Luc}(n) + \sqrt{5 \cdot (\text{Luc}(n)^2 - 4 \cdot (-1)^n)}}{2}$ .
- (37) For every natural number  $n$  such that  $n \geq 4$  holds  $\text{Luc}(n + 1) = \lfloor \frac{\text{Luc}(n) + 1 + \sqrt{(5 \cdot \text{Luc}(n)^2 - 2 \cdot \text{Luc}(n)) + 1}}{2} \rfloor$ .
- (38) For every natural number  $n$  such that  $n > 2$  holds  $\text{Luc}(n) = \lfloor \frac{1}{\tau} \cdot (\text{Luc}(n + 1) + \frac{1}{2}) \rfloor$ .
- (39) For all natural numbers  $n, k$  such that  $n \geq 4$  and  $k \geq 1$  and  $n > k$  and  $n$  is odd holds  $\text{Luc}(n + k) = \lfloor \tau^k \cdot \text{Luc}(n) + 1 \rfloor$ .

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