

Representation of the Fibonacci and Lucas Numbers in Terms of Floor and Ceiling

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Summary. In the paper we show how to express the Fibonacci numbers and Lucas numbers using the floor and ceiling operations.

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The notation and terminology used here have been introduced in the following papers: [7], [3], [8], [11], [10], [1], [4], [6], [2], [5], and [9].

1. PRELIMINARIES

One can prove the following propositions:

- (1) For all real numbers a, b and for every natural number c holds $(\frac{a}{b})^c = \frac{a^c}{b^c}$.
- (2) For every real number a and for all integer numbers b, c such that $a \neq 0$ holds $a^{b+c} = a^b \cdot a^c$.
- (3) For every natural number n and for every real number a such that n is even and $a \neq 0$ holds $(-a)^n = a^n$.
- (4) For every natural number n and for every real number a such that n is odd and $a \neq 0$ holds $(-a)^n = -a^n$.
- (5) $|\bar{\tau}| < 1$.
- (6) For every natural number n and for every non empty real number r such that n is even holds $r^n > 0$.
- (7) For every natural number n and for every real number r such that n is odd and $r < 0$ holds $r^n < 0$.

- (8) For every natural number n such that $n \neq 0$ holds $\bar{\tau}^n < \frac{1}{2}$.
- (9) For all natural numbers n, m and for every real number r such that m is odd and $n \geq m$ and $r < 0$ and $r > -1$ holds $r^n \geq r^m$.
- (10) For all natural numbers n, m such that m is odd and $n \geq m$ holds $\bar{\tau}^n \geq \bar{\tau}^m$.
- (11) For all natural numbers n, m such that n is even and m is even and $n \geq m$ holds $\bar{\tau}^n \leq \bar{\tau}^m$.
- (12) For all non empty natural numbers m, n such that $m \geq n$ holds $\text{Luc}(m) \geq \text{Luc}(n)$.
- (13) For every non empty natural number n holds $\tau^n > \bar{\tau}^n$.
- (14) For every natural number n such that $n > 1$ holds $-\frac{1}{2} < \bar{\tau}^n$.
- (15) For every natural number n such that $n > 2$ holds $\bar{\tau}^n \geq -\frac{1}{\sqrt{5}}$.
- (16) For every natural number n such that $n \geq 2$ holds $\bar{\tau}^n \leq \frac{1}{\sqrt{5}}$.
- (17) For every natural number n holds $\frac{\bar{\tau}^n}{\sqrt{5}} + \frac{1}{2} > 0$ and $\frac{\bar{\tau}^n}{\sqrt{5}} + \frac{1}{2} < 1$.

2. FORMULAS FOR THE FIBONACCI NUMBERS

Next we state two propositions:

- (18) For every natural number n holds $\lfloor \frac{\tau^n}{\sqrt{5}} + \frac{1}{2} \rfloor = \text{Fib}(n)$.
- (19) For every natural number n such that $n \neq 0$ holds $\lceil \frac{\tau^n}{\sqrt{5}} - \frac{1}{2} \rceil = \text{Fib}(n)$.

We now state a number of propositions:

- (20) For every natural number n such that $n \neq 0$ holds $\lfloor \frac{\tau^{2 \cdot n}}{\sqrt{5}} \rfloor = \text{Fib}(2 \cdot n)$.
- (21) For every natural number n holds $\lceil \frac{\tau^{2 \cdot n + 1}}{\sqrt{5}} \rceil = \text{Fib}(2 \cdot n + 1)$.
- (22) For every natural number n such that $n \geq 2$ and n is even holds $\text{Fib}(n + 1) = \lfloor \tau \cdot \text{Fib}(n) + 1 \rfloor$.
- (23) For every natural number n such that $n \geq 2$ and n is odd holds $\text{Fib}(n + 1) = \lceil \tau \cdot \text{Fib}(n) - 1 \rceil$.
- (24) For every natural number n such that $n \geq 2$ holds $\text{Fib}(n + 1) = \lfloor \frac{\text{Fib}(n) + \sqrt{5} \cdot \text{Fib}(n) + 1}{2} \rfloor$.
- (25) For every natural number n such that $n \geq 2$ holds $\text{Fib}(n + 1) = \lceil \frac{(\text{Fib}(n) + \sqrt{5} \cdot \text{Fib}(n)) - 1}{2} \rceil$.
- (26) For every natural number n holds $\text{Fib}(n + 1) = \frac{\text{Fib}(n) + \sqrt{5 \cdot \text{Fib}(n)^2 + 4 \cdot (-1)^n}}{2}$.
- (27) For every natural number n such that $n \geq 2$ holds $\text{Fib}(n + 1) = \lfloor \frac{\text{Fib}(n) + 1 + \sqrt{(5 \cdot \text{Fib}(n)^2 - 2 \cdot \text{Fib}(n)) + 1}}{2} \rfloor$.
- (28) For every natural number n such that $n \geq 2$ holds $\text{Fib}(n) = \lfloor \frac{1}{\tau} \cdot (\text{Fib}(n + 1) + \frac{1}{2}) \rfloor$.

- (29) For all natural numbers n, k such that $n \geq k > 1$ or $k = 1$ and $n > k$ holds $\lfloor \tau^k \cdot \text{Fib}(n) + \frac{1}{2} \rfloor = \text{Fib}(n + k)$.

3. FORMULAS FOR THE LUCAS NUMBERS

Next we state a number of propositions:

- (30) For every natural number n such that $n \geq 2$ holds $\text{Luc}(n) = \lfloor \tau^n + \frac{1}{2} \rfloor$.
- (31) For every natural number n such that $n \geq 2$ holds $\text{Luc}(n) = \lceil \tau^n - \frac{1}{2} \rceil$.
- (32) For every natural number n such that $n \geq 2$ holds $\text{Luc}(2 \cdot n) = \lceil \tau^{2 \cdot n} \rceil$.
- (33) For every natural number n such that $n \geq 2$ holds $\text{Luc}(2 \cdot n + 1) = \lfloor \tau^{2 \cdot n + 1} \rfloor$.
- (34) For every natural number n such that $n \geq 2$ and n is odd holds $\text{Luc}(n + 1) = \lfloor \tau \cdot \text{Luc}(n) + 1 \rfloor$.
- (35) For every natural number n such that $n \geq 2$ and n is even holds $\text{Luc}(n + 1) = \lceil \tau \cdot \text{Luc}(n) - 1 \rceil$.
- (36) For every natural number n such that $n \neq 1$ holds $\text{Luc}(n + 1) = \frac{\text{Luc}(n) + \sqrt{5 \cdot (\text{Luc}(n)^2 - 4 \cdot (-1)^n)}}{2}$.
- (37) For every natural number n such that $n \geq 4$ holds $\text{Luc}(n + 1) = \lfloor \frac{\text{Luc}(n) + 1 + \sqrt{(5 \cdot \text{Luc}(n)^2 - 2 \cdot \text{Luc}(n)) + 1}}{2} \rfloor$.
- (38) For every natural number n such that $n > 2$ holds $\text{Luc}(n) = \lfloor \frac{1}{\tau} \cdot (\text{Luc}(n + 1) + \frac{1}{2}) \rfloor$.
- (39) For all natural numbers n, k such that $n \geq 4$ and $k \geq 1$ and $n > k$ and n is odd holds $\text{Luc}(n + k) = \lfloor \tau^k \cdot \text{Luc}(n) + 1 \rfloor$.

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