

# The Correspondence Between $n$ -dimensional Euclidean Space and the Product of $n$ Real Lines

Artur Kornilowicz  
Institute of Informatics  
University of Białystok  
Sosnowa 64, 15-887 Białystok, Poland

**Summary.** In the article we prove that a family of open  $n$ -hypercubes is a basis of  $n$ -dimensional Euclidean space. The equality of the space and the product of  $n$  real lines has been proven.

MML identifier: EUCLID\_9, version: 7.11.05 4.134.1080

The terminology and notation used in this paper have been introduced in the following papers: [2], [6], [10], [4], [7], [18], [8], [13], [1], [3], [5], [15], [16], [17], [21], [22], [9], [19], [20], [11], [14], and [12].

For simplicity, we use the following convention:  $x, y$  are sets,  $i, n$  are natural numbers,  $r, s$  are real numbers, and  $f_1, f_2$  are  $n$ -long real-valued finite sequences.

Let  $s$  be a real number and let  $r$  be a non positive real number. One can check the following observations:

- \*  $]s - r, s + r[$  is empty,
- \*  $[s - r, s + r[$  is empty, and
- \*  $]s - r, s + r]$  is empty.

Let  $s$  be a real number and let  $r$  be a negative real number. Observe that  $[s - r, s + r]$  is empty.

Let  $f$  be an empty yielding function and let us consider  $x$ . Observe that  $f(x)$  is empty.

Let us consider  $i$ . Observe that  $i \mapsto 0$  is empty yielding.

Let  $f$  be an  $n$ -long complex-valued finite sequence. One can check the following observations:

- \*  $-f$  is  $n$ -long,
- \*  $f^{-1}$  is  $n$ -long,
- \*  $f^2$  is  $n$ -long, and
- \*  $|f|$  is  $n$ -long.

Let  $g$  be an  $n$ -long complex-valued finite sequence. One can verify the following observations:

- \*  $f + g$  is  $n$ -long,
- \*  $f - g$  is  $n$ -long,
- \*  $f g$  is  $n$ -long, and
- \*  $f/g$  is  $n$ -long.

Let  $c$  be a complex number and let  $f$  be an  $n$ -long complex-valued finite sequence. One can check the following observations:

- \*  $c + f$  is  $n$ -long,
- \*  $f - c$  is  $n$ -long, and
- \*  $c f$  is  $n$ -long.

Let  $f$  be a real-valued function. Note that  $\{f\}$  is real-functions-membered. Let  $g$  be a real-valued function. One can verify that  $\{f, g\}$  is real-functions-membered.

Let  $D$  be a set and let us consider  $n$ . Note that  $D^n$  is finite sequence-membered.

Let us consider  $n$ . Note that  $\mathcal{R}^n$  is finite sequence-membered.

Let us consider  $n$ . Observe that  $\mathcal{R}^n$  is real-functions-membered.

Let us consider  $x, y$  and let  $f$  be an  $n$ -long finite sequence. Observe that  $f + \cdot (x, y)$  is  $n$ -long.

One can prove the following three propositions:

- (1) For every  $n$ -long finite sequence  $f$  such that  $f$  is empty holds  $n = 0$ .
- (2) For every  $n$ -long real-valued finite sequence  $f$  holds  $f \in \mathcal{R}^n$ .
- (3) For all complex-valued functions  $f, g$  holds  $|f - g| = |g - f|$ .

Let us consider  $f_1, f_2$ . The functor  $\text{max-diff-index}(f_1, f_2)$  yields a natural number and is defined as follows:

(Def. 1)  $\text{max-diff-index}(f_1, f_2)$  is the element of  $|f_1 - f_2|^{-1}(\{\sup \text{rng}|f_1 - f_2|\})$ .

Let us note that the functor  $\text{max-diff-index}(f_1, f_2)$  is commutative.

One can prove the following propositions:

- (4) If  $n \neq 0$ , then  $\text{max-diff-index}(f_1, f_2) \in \text{dom } f_1$ .
- (5)  $|f_1 - f_2|(x) \leq |f_1 - f_2|(\text{max-diff-index}(f_1, f_2))$ .

One can verify that the metric space of real numbers is real-membered.

Let us observe that  $(\mathcal{E}^0)_{\text{top}}$  is trivial.

Let us consider  $n$ . Observe that  $\mathcal{E}^n$  is constituted finite sequences.

Let us consider  $n$ . One can verify that every point of  $\mathcal{E}^n$  is real-valued.

Let us consider  $n$ . One can check that every point of  $\mathcal{E}^n$  is  $n$ -long.

The following two propositions are true:

- (6) The open set family of  $\mathcal{E}^0 = \{\emptyset, \{\emptyset\}\}$ .
- (7) For every subset  $B$  of  $\mathcal{E}^0$  holds  $B = \emptyset$  or  $B = \{\emptyset\}$ .

In the sequel  $e, e_1$  are points of  $\mathcal{E}^n$ .

Let us consider  $n, e$ . The functor  ${}^@e$  yields a point of  $(\mathcal{E}^n)_{\text{top}}$  and is defined by:

(Def. 2)  ${}^@e = e$ .

Let us consider  $n, e$  and let  $r$  be a non positive real number. Observe that  $\text{Ball}(e, r)$  is empty.

Let us consider  $n, e$  and let  $r$  be a positive real number. Note that  $\text{Ball}(e, r)$  is non empty.

We now state three propositions:

- (8) For all points  $p_1, p_2$  of  $\mathcal{E}_{\mathbb{T}}^n$  such that  $i \in \text{dom } p_1$  holds  $(p_1(i) - p_2(i))^2 \leq \sum^2(p_1 - p_2)$ .
- (9) Let  $n$  be an element of  $\mathbb{N}$  and  $a, o, p$  be elements of  $\mathcal{E}_{\mathbb{T}}^n$ . If  $a \in \text{Ball}(o, r)$ , then for every set  $x$  holds  $|(a - o)(x)| < r$  and  $|a(x) - o(x)| < r$ .
- (10) For all points  $a, o$  of  $\mathcal{E}^n$  such that  $a \in \text{Ball}(o, r)$  and for every set  $x$  holds  $|(a - o)(x)| < r$  and  $|a(x) - o(x)| < r$ .

Let  $f$  be a real-valued function and let  $r$  be a real number. The functor  $\text{Intervals}(f, r)$  yields a function and is defined as follows:

(Def. 3)  $\text{dom Intervals}(f, r) = \text{dom } f$  and for every set  $x$  such that  $x \in \text{dom } f$  holds  $(\text{Intervals}(f, r))(x) = ]f(x) - r, f(x) + r[$ .

Let us consider  $r$ . Note that  $\text{Intervals}(\emptyset, r)$  is empty.

Let  $f$  be a real-valued finite sequence and let us consider  $r$ . One can check that  $\text{Intervals}(f, r)$  is finite sequence-like.

Let us consider  $n, e, r$ . The functor  $\text{OpenHypercube}(e, r)$  yielding a subset of  $(\mathcal{E}^n)_{\text{top}}$  is defined by:

(Def. 4)  $\text{OpenHypercube}(e, r) = \coprod \text{Intervals}(e, r)$ .

Next we state the proposition

- (11) If  $0 < r$ , then  $e \in \text{OpenHypercube}(e, r)$ .

Let  $n$  be a non zero natural number, let  $e$  be a point of  $\mathcal{E}^n$ , and let  $r$  be a non positive real number. Observe that  $\text{OpenHypercube}(e, r)$  is empty.

One can prove the following proposition

- (12) For every point  $e$  of  $\mathcal{E}^0$  holds  $\text{OpenHypercube}(e, r) = \{\emptyset\}$ .

Let  $e$  be a point of  $\mathcal{E}^0$  and let us consider  $r$ . Note that  $\text{OpenHypercube}(e, r)$  is non empty.

Let us consider  $n, e$  and let  $r$  be a positive real number. One can check that  $\text{OpenHypercube}(e, r)$  is non empty.

One can prove the following propositions:

- (13) If  $r \leq s$ , then  $\text{OpenHypercube}(e, r) \subseteq \text{OpenHypercube}(e, s)$ .
- (14) If  $n \neq 0$  or  $0 < r$  and if  $e_1 \in \text{OpenHypercube}(e, r)$ , then for every set  $x$  holds  $|(e_1 - e)(x)| < r$  and  $|e_1(x) - e(x)| < r$ .
- (15) If  $n \neq 0$  and  $e_1 \in \text{OpenHypercube}(e, r)$ , then  $\sum^2(e_1 - e) < n \cdot r^2$ .
- (16) If  $n \neq 0$  and  $e_1 \in \text{OpenHypercube}(e, r)$ , then  $\rho(e_1, e) < r \cdot \sqrt{n}$ .
- (17) If  $n \neq 0$ , then  $\text{OpenHypercube}(e, \frac{r}{\sqrt{n}}) \subseteq \text{Ball}(e, r)$ .
- (18) If  $n \neq 0$ , then  $\text{OpenHypercube}(e, r) \subseteq \text{Ball}(e, r \cdot \sqrt{n})$ .
- (19) If  $e_1 \in \text{Ball}(e, r)$ , then there exists a non zero element  $m$  of  $\mathbb{N}$  such that  $\text{OpenHypercube}(e_1, \frac{1}{m}) \subseteq \text{Ball}(e, r)$ .
- (20) If  $n \neq 0$  and  $e_1 \in \text{OpenHypercube}(e, r)$ , then  $r > |e_1 - e|(\text{max-diff-index}(e_1, e))$ .
- (21)  $\text{OpenHypercube}(e_1, r - |e_1 - e|(\text{max-diff-index}(e_1, e))) \subseteq \text{OpenHypercube}(e, r)$ .
- (22)  $\text{Ball}(e, r) \subseteq \text{OpenHypercube}(e, r)$ .

Let us consider  $n, e, r$ . Observe that  $\text{OpenHypercube}(e, r)$  is open.

We now state two propositions:

- (23) Let  $V$  be a subset of  $(\mathcal{E}^n)_{\text{top}}$ . Suppose  $V$  is open. Let  $e$  be a point of  $\mathcal{E}^n$ . If  $e \in V$ , then there exists a non zero element  $m$  of  $\mathbb{N}$  such that  $\text{OpenHypercube}(e, \frac{1}{m}) \subseteq V$ .
- (24) Let  $V$  be a subset of  $(\mathcal{E}^n)_{\text{top}}$ . Suppose that for every point  $e$  of  $\mathcal{E}^n$  such that  $e \in V$  there exists a real number  $r$  such that  $r > 0$  and  $\text{OpenHypercube}(e, r) \subseteq V$ . Then  $V$  is open.

Let us consider  $n, e$ . The functor  $\text{OpenHypercubes } e$  yields a family of subsets of  $(\mathcal{E}^n)_{\text{top}}$  and is defined by:

- (Def. 5)  $\text{OpenHypercubes } e = \{\text{OpenHypercube}(e, \frac{1}{m}) : m \text{ ranges over non zero elements of } \mathbb{N}\}$ .

Let us consider  $n, e$ . Observe that  $\text{OpenHypercubes } e$  is non empty, open, and  $e$ -quasi-basis.

Next we state four propositions:

- (25) For every 1-sorted yielding many sorted set  $J$  indexed by  $\text{Seg } n$  such that  $J = \text{Seg } n \mapsto \mathbb{R}^1$  holds  $\mathbb{R}^{\text{Seg } n} = \prod (\text{the support of } J)$ .
- (26) Let  $J$  be a topological space yielding many sorted set indexed by  $\text{Seg } n$ . Suppose  $n \neq 0$  and  $J = \text{Seg } n \mapsto \mathbb{R}^1$ . Let  $P_1$  be a family of subsets of  $(\mathcal{E}^n)_{\text{top}}$ . If  $P_1$  is the product prebasis for  $J$ , then  $P_1$  is quasi-prebasis.
- (27) Let  $J$  be a topological space yielding many sorted set indexed by  $\text{Seg } n$ . Suppose  $J = \text{Seg } n \mapsto \mathbb{R}^1$ . Let  $P_1$  be a family of subsets of  $(\mathcal{E}^n)_{\text{top}}$ . If  $P_1$  is the product prebasis for  $J$ , then  $P_1$  is open.
- (28)  $(\mathcal{E}^n)_{\text{top}} = \prod (\text{Seg } n \mapsto \mathbb{R}^1)$ .

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. König’s theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [5] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(4):481–485, 1991.
- [6] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(4):661–668, 1990.
- [10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [11] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(4):605–608, 1991.
- [12] Jarosław Gryko. Injective spaces. *Formalized Mathematics*, 7(1):57–62, 1998.
- [13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(3):607–610, 1990.
- [14] Artur Korniłowicz. Arithmetic operations on functions from sets into functional sets. *Formalized Mathematics*, 17(1):43–60, 2009, doi:10.2478/v10037-009-0005-y.
- [15] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(1):103–108, 1993.
- [16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [17] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

Received November 30, 2009

---