

The Correspondence Between n -dimensional Euclidean Space and the Product of n Real Lines

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Summary. In the article we prove that a family of open n -hypercubes is a basis of n -dimensional Euclidean space. The equality of the space and the product of n real lines has been proven.

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The terminology and notation used in this paper have been introduced in the following papers: [2], [6], [10], [4], [7], [18], [8], [13], [1], [3], [5], [15], [16], [17], [21], [22], [9], [19], [20], [11], [14], and [12].

For simplicity, we use the following convention: x, y are sets, i, n are natural numbers, r, s are real numbers, and f_1, f_2 are n -long real-valued finite sequences.

Let s be a real number and let r be a non positive real number. One can check the following observations:

- * $]s - r, s + r[$ is empty,
- * $[s - r, s + r[$ is empty, and
- * $]s - r, s + r]$ is empty.

Let s be a real number and let r be a negative real number. Observe that $[s - r, s + r]$ is empty.

Let f be an empty yielding function and let us consider x . Observe that $f(x)$ is empty.

Let us consider i . Observe that $i \mapsto 0$ is empty yielding.

Let f be an n -long complex-valued finite sequence. One can check the following observations:

- * $-f$ is n -long,
- * f^{-1} is n -long,
- * f^2 is n -long, and
- * $|f|$ is n -long.

Let g be an n -long complex-valued finite sequence. One can verify the following observations:

- * $f + g$ is n -long,
- * $f - g$ is n -long,
- * $f g$ is n -long, and
- * f/g is n -long.

Let c be a complex number and let f be an n -long complex-valued finite sequence. One can check the following observations:

- * $c + f$ is n -long,
- * $f - c$ is n -long, and
- * $c f$ is n -long.

Let f be a real-valued function. Note that $\{f\}$ is real-functions-membered. Let g be a real-valued function. One can verify that $\{f, g\}$ is real-functions-membered.

Let D be a set and let us consider n . Note that D^n is finite sequence-membered.

Let us consider n . Note that \mathcal{R}^n is finite sequence-membered.

Let us consider n . Observe that \mathcal{R}^n is real-functions-membered.

Let us consider x, y and let f be an n -long finite sequence. Observe that $f + \cdot (x, y)$ is n -long.

One can prove the following three propositions:

- (1) For every n -long finite sequence f such that f is empty holds $n = 0$.
- (2) For every n -long real-valued finite sequence f holds $f \in \mathcal{R}^n$.
- (3) For all complex-valued functions f, g holds $|f - g| = |g - f|$.

Let us consider f_1, f_2 . The functor $\text{max-diff-index}(f_1, f_2)$ yields a natural number and is defined as follows:

(Def. 1) $\text{max-diff-index}(f_1, f_2)$ is the element of $|f_1 - f_2|^{-1}(\{\sup \text{rng}|f_1 - f_2|\})$.

Let us note that the functor $\text{max-diff-index}(f_1, f_2)$ is commutative.

One can prove the following propositions:

- (4) If $n \neq 0$, then $\text{max-diff-index}(f_1, f_2) \in \text{dom } f_1$.
- (5) $|f_1 - f_2|(x) \leq |f_1 - f_2|(\text{max-diff-index}(f_1, f_2))$.

One can verify that the metric space of real numbers is real-membered.

Let us observe that $(\mathcal{E}^0)_{\text{top}}$ is trivial.

Let us consider n . Observe that \mathcal{E}^n is constituted finite sequences.

Let us consider n . One can verify that every point of \mathcal{E}^n is real-valued.

Let us consider n . One can check that every point of \mathcal{E}^n is n -long.

The following two propositions are true:

- (6) The open set family of $\mathcal{E}^0 = \{\emptyset, \{\emptyset\}\}$.
- (7) For every subset B of \mathcal{E}^0 holds $B = \emptyset$ or $B = \{\emptyset\}$.

In the sequel e, e_1 are points of \mathcal{E}^n .

Let us consider n, e . The functor ${}^{\textcircled{a}}e$ yields a point of $(\mathcal{E}^n)_{\text{top}}$ and is defined by:

(Def. 2) ${}^{\textcircled{a}}e = e$.

Let us consider n, e and let r be a non positive real number. Observe that $\text{Ball}(e, r)$ is empty.

Let us consider n, e and let r be a positive real number. Note that $\text{Ball}(e, r)$ is non empty.

We now state three propositions:

- (8) For all points p_1, p_2 of $\mathcal{E}_{\mathbb{T}}^n$ such that $i \in \text{dom } p_1$ holds $(p_1(i) - p_2(i))^2 \leq \sum^2(p_1 - p_2)$.
- (9) Let n be an element of \mathbb{N} and a, o, p be elements of $\mathcal{E}_{\mathbb{T}}^n$. If $a \in \text{Ball}(o, r)$, then for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.
- (10) For all points a, o of \mathcal{E}^n such that $a \in \text{Ball}(o, r)$ and for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.

Let f be a real-valued function and let r be a real number. The functor $\text{Intervals}(f, r)$ yields a function and is defined as follows:

(Def. 3) $\text{dom Intervals}(f, r) = \text{dom } f$ and for every set x such that $x \in \text{dom } f$ holds $(\text{Intervals}(f, r))(x) =]f(x) - r, f(x) + r[$.

Let us consider r . Note that $\text{Intervals}(\emptyset, r)$ is empty.

Let f be a real-valued finite sequence and let us consider r . One can check that $\text{Intervals}(f, r)$ is finite sequence-like.

Let us consider n, e, r . The functor $\text{OpenHypercube}(e, r)$ yielding a subset of $(\mathcal{E}^n)_{\text{top}}$ is defined by:

(Def. 4) $\text{OpenHypercube}(e, r) = \coprod \text{Intervals}(e, r)$.

Next we state the proposition

- (11) If $0 < r$, then $e \in \text{OpenHypercube}(e, r)$.

Let n be a non zero natural number, let e be a point of \mathcal{E}^n , and let r be a non positive real number. Observe that $\text{OpenHypercube}(e, r)$ is empty.

One can prove the following proposition

- (12) For every point e of \mathcal{E}^0 holds $\text{OpenHypercube}(e, r) = \{\emptyset\}$.

Let e be a point of \mathcal{E}^0 and let us consider r . Note that $\text{OpenHypercube}(e, r)$ is non empty.

Let us consider n, e and let r be a positive real number. One can check that $\text{OpenHypercube}(e, r)$ is non empty.

One can prove the following propositions:

- (13) If $r \leq s$, then $\text{OpenHypercube}(e, r) \subseteq \text{OpenHypercube}(e, s)$.
- (14) If $n \neq 0$ or $0 < r$ and if $e_1 \in \text{OpenHypercube}(e, r)$, then for every set x holds $|(e_1 - e)(x)| < r$ and $|e_1(x) - e(x)| < r$.
- (15) If $n \neq 0$ and $e_1 \in \text{OpenHypercube}(e, r)$, then $\sum^2(e_1 - e) < n \cdot r^2$.
- (16) If $n \neq 0$ and $e_1 \in \text{OpenHypercube}(e, r)$, then $\rho(e_1, e) < r \cdot \sqrt{n}$.
- (17) If $n \neq 0$, then $\text{OpenHypercube}(e, \frac{r}{\sqrt{n}}) \subseteq \text{Ball}(e, r)$.
- (18) If $n \neq 0$, then $\text{OpenHypercube}(e, r) \subseteq \text{Ball}(e, r \cdot \sqrt{n})$.
- (19) If $e_1 \in \text{Ball}(e, r)$, then there exists a non zero element m of \mathbb{N} such that $\text{OpenHypercube}(e_1, \frac{1}{m}) \subseteq \text{Ball}(e, r)$.
- (20) If $n \neq 0$ and $e_1 \in \text{OpenHypercube}(e, r)$, then $r > |e_1 - e|(\text{max-diff-index}(e_1, e))$.
- (21) $\text{OpenHypercube}(e_1, r - |e_1 - e|(\text{max-diff-index}(e_1, e))) \subseteq \text{OpenHypercube}(e, r)$.
- (22) $\text{Ball}(e, r) \subseteq \text{OpenHypercube}(e, r)$.

Let us consider n, e, r . Observe that $\text{OpenHypercube}(e, r)$ is open.

We now state two propositions:

- (23) Let V be a subset of $(\mathcal{E}^n)_{\text{top}}$. Suppose V is open. Let e be a point of \mathcal{E}^n . If $e \in V$, then there exists a non zero element m of \mathbb{N} such that $\text{OpenHypercube}(e, \frac{1}{m}) \subseteq V$.
- (24) Let V be a subset of $(\mathcal{E}^n)_{\text{top}}$. Suppose that for every point e of \mathcal{E}^n such that $e \in V$ there exists a real number r such that $r > 0$ and $\text{OpenHypercube}(e, r) \subseteq V$. Then V is open.

Let us consider n, e . The functor $\text{OpenHypercubes } e$ yields a family of subsets of $(\mathcal{E}^n)_{\text{top}}$ and is defined by:

- (Def. 5) $\text{OpenHypercubes } e = \{\text{OpenHypercube}(e, \frac{1}{m}) : m \text{ ranges over non zero elements of } \mathbb{N}\}$.

Let us consider n, e . Observe that $\text{OpenHypercubes } e$ is non empty, open, and e -quasi-basis.

Next we state four propositions:

- (25) For every 1-sorted yielding many sorted set J indexed by $\text{Seg } n$ such that $J = \text{Seg } n \mapsto \mathbb{R}^1$ holds $\mathbb{R}^{\text{Seg } n} = \prod (\text{the support of } J)$.
- (26) Let J be a topological space yielding many sorted set indexed by $\text{Seg } n$. Suppose $n \neq 0$ and $J = \text{Seg } n \mapsto \mathbb{R}^1$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\text{top}}$. If P_1 is the product prebasis for J , then P_1 is quasi-prebasis.
- (27) Let J be a topological space yielding many sorted set indexed by $\text{Seg } n$. Suppose $J = \text{Seg } n \mapsto \mathbb{R}^1$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\text{top}}$. If P_1 is the product prebasis for J , then P_1 is open.
- (28) $(\mathcal{E}^n)_{\text{top}} = \prod (\text{Seg } n \mapsto \mathbb{R}^1)$.

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