

# The Sum and Product of Finite Sequences of Complex Numbers

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**Summary.** This article extends the [10]. We define the sum and the product of the sequence of complex numbers, and formalize these theorems. Our method refers to the [11].

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The notation and terminology used in this paper have been introduced in the following papers: [5], [7], [6], [4], [8], [13], [9], [2], [3], [15], [10], [12], and [14].

## 1. AUXILIARY THEOREMS

Let  $F$  be a complex-valued binary relation. Then  $\text{rng } F$  is a subset of  $\mathbb{C}$ .

Let  $D$  be a non empty set, let  $F$  be a function from  $\mathbb{C}$  into  $D$ , and let  $F_1$  be a complex-valued finite sequence. Note that  $F \cdot F_1$  is finite sequence-like.

For simplicity, we adopt the following rules:  $i, j$  denote natural numbers,  $x, x_1$  denote elements of  $\mathbb{C}$ ,  $c$  denotes a complex number,  $F, F_1, F_2$  denote complex-valued finite sequences, and  $R, R_1$  denote  $i$ -element finite sequences of elements of  $\mathbb{C}$ .

The unary operation  $\text{sqrcomplex}$  on  $\mathbb{C}$  is defined as follows:

(Def. 1) For every  $c$  holds  $(\text{sqrcomplex})(c) = c^2$ .

Next we state two propositions:

- (1)  $\text{sqrcomplex}$  is distributive w.r.t.  $\cdot_{\mathbb{C}}$ .
- (2)  $\cdot_{\mathbb{C}}$  is distributive w.r.t.  $+_{\mathbb{C}}$ .

2. SOME FUNCTORS ON THE  $i$ -TUPLES OF COMPLEX NUMBERS

Let us consider  $F_1, F_2$ . Then  $F_1 + F_2$  is a finite sequence of elements of  $\mathbb{C}$  and it can be characterized by the condition:

$$\text{(Def. 2)} \quad F_1 + F_2 = (+_{\mathbb{C}})^{\circ}(F_1, F_2).$$

Let us observe that the functor  $F_1 + F_2$  is commutative.

Let us consider  $i, R_1, R_2$ . Then  $R_1 + R_2$  is an element of  $\mathbb{C}^i$ .

The following propositions are true:

$$(3) \quad (R_1 + R_2)(s) = R_1(s) + R_2(s).$$

$$(4) \quad \varepsilon_{\mathbb{C}} + F = \varepsilon_{\mathbb{C}}.$$

$$(5) \quad \langle c_1 \rangle + \langle c_2 \rangle = \langle c_1 + c_2 \rangle.$$

$$(6) \quad i \mapsto c_1 + i \mapsto c_2 = i \mapsto (c_1 + c_2).$$

Let us consider  $F$ . Then  $-F$  is a finite sequence of elements of  $\mathbb{C}$  and it can be characterized by the condition:

$$\text{(Def. 3)} \quad -F = -_{\mathbb{C}} \cdot F.$$

Let us consider  $i, R$ . Then  $-R$  is an element of  $\mathbb{C}^i$ .

The following propositions are true:

$$(7) \quad -\langle c \rangle = \langle -c \rangle.$$

$$(8) \quad -i \mapsto c = i \mapsto (-c).$$

$$(9) \quad \text{If } R_1 + R = R_2 + R, \text{ then } R_1 = R_2.$$

$$(10) \quad -(F_1 + F_2) = -F_1 + -F_2.$$

Let us consider  $F_1, F_2$ . Then  $F_1 - F_2$  is a finite sequence of elements of  $\mathbb{C}$  and it can be characterized by the condition:

$$\text{(Def. 4)} \quad F_1 - F_2 = (-_{\mathbb{C}})^{\circ}(F_1, F_2).$$

Let us consider  $i, R_1, R_2$ . Then  $R_1 - R_2$  is an element of  $\mathbb{C}^i$ .

The following propositions are true:

$$(11) \quad (R_1 - R_2)(s) = R_1(s) - R_2(s).$$

$$(12) \quad \varepsilon_{\mathbb{C}} - F = \varepsilon_{\mathbb{C}} \text{ and } F - \varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{C}}.$$

$$(13) \quad \langle c_1 \rangle - \langle c_2 \rangle = \langle c_1 - c_2 \rangle.$$

$$(14) \quad i \mapsto c_1 - i \mapsto c_2 = i \mapsto (c_1 - c_2).$$

$$(15) \quad R - i \mapsto 0_{\mathbb{C}} = R.$$

$$(16) \quad -(F_1 - F_2) = F_2 - F_1.$$

$$(17) \quad -(F_1 - F_2) = -F_1 + F_2.$$

$$(18) \quad \text{If } R_1 - R_2 = i \mapsto 0_{\mathbb{C}}, \text{ then } R_1 = R_2.$$

$$(19) \quad R_1 = (R_1 + R) - R.$$

$$(20) \quad R_1 = (R_1 - R) + R.$$

Let us consider  $F, c$ . We introduce  $c \cdot F$  as a synonym of  $cF$ .

Let us consider  $F, c$ . Then  $c \cdot F$  is a finite sequence of elements of  $\mathbb{C}$  and it can be characterized by the condition:

(Def. 5)  $c \cdot F = \cdot_c \cdot F$ .

Let us consider  $i, R, c$ . Then  $c \cdot R$  is an element of  $\mathbb{C}^i$ .

One can prove the following four propositions:

(21)  $c \cdot \langle c_1 \rangle = \langle c \cdot c_1 \rangle$ .

(22)  $c_1 \cdot (i \mapsto c_2) = i \mapsto (c_1 \cdot c_2)$ .

(23)  $(c_1 + c_2) \cdot F = c_1 \cdot F + c_2 \cdot F$ .

(24)  $0_{\mathbb{C}} \cdot R = i \mapsto 0_{\mathbb{C}}$ .

Let us consider  $F_1, F_2$ . We introduce  $F_1 \bullet F_2$  as a synonym of  $F_1 F_2$ .

Let us consider  $F_1, F_2$ . Then  $F_1 \bullet F_2$  is a finite sequence of elements of  $\mathbb{C}$  and it can be characterized by the condition:

(Def. 6)  $F_1 \bullet F_2 = (\cdot_{\mathbb{C}})^{\circ}(F_1, F_2)$ .

Let us note that the functor  $F_1 \bullet F_2$  is commutative.

Let us consider  $i, R_1, R_2$ . Then  $R_1 \bullet R_2$  is an element of  $\mathbb{C}^i$ .

Next we state four propositions:

(25)  $\varepsilon_{\mathbb{C}} \bullet F = \varepsilon_{\mathbb{C}}$ .

(26)  $\langle c_1 \rangle \bullet \langle c_2 \rangle = \langle c_1 \cdot c_2 \rangle$ .

(27)  $i \mapsto c \bullet R = c \cdot R$ .

(28)  $i \mapsto c_1 \bullet i \mapsto c_2 = i \mapsto (c_1 \cdot c_2)$ .

### 3. FINITE SUM OF FINITE SEQUENCE OF COMPLEX NUMBERS

One can prove the following propositions:

(29)  $\sum(\varepsilon_{\mathbb{C}}) = 0_{\mathbb{C}}$ .

(30)  $\sum\langle c \rangle = c$ .

(31)  $\sum(F \wedge \langle c \rangle) = \sum F + c$ .

(32)  $\sum(F_1 \wedge F_2) = \sum F_1 + \sum F_2$ .

(33)  $\sum(\langle c \rangle \wedge F) = c + \sum F$ .

(34)  $\sum\langle c_1, c_2 \rangle = c_1 + c_2$ .

(35)  $\sum\langle c_1, c_2, c_3 \rangle = c_1 + c_2 + c_3$ .

(36)  $\sum(i \mapsto c) = i \cdot c$ .

(37)  $\sum(i \mapsto 0_{\mathbb{C}}) = 0_{\mathbb{C}}$ .

(38)  $\sum(c \cdot F) = c \cdot \sum F$ .

(39)  $\sum(-F) = -\sum F$ .

(40)  $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$ .

(41)  $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$ .

## 4. THE PRODUCT OF FINITE SEQUENCES OF COMPLEX NUMBERS

One can prove the following propositions:

- (42)  $\prod(\varepsilon_{\mathbb{C}}) = 1.$
- (43)  $\prod(\langle c \rangle \wedge F) = c \cdot \prod F.$
- (44) For every element  $R$  of  $\mathbb{C}^0$  holds  $\prod R = 1.$
- (45)  $\prod((i + j) \mapsto c) = \prod(i \mapsto c) \cdot \prod(j \mapsto c).$
- (46)  $\prod((i \cdot j) \mapsto c) = \prod(j \mapsto \prod(i \mapsto c)).$
- (47)  $\prod(i \mapsto (c_1 \cdot c_2)) = \prod(i \mapsto c_1) \cdot \prod(i \mapsto c_2).$
- (48)  $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2.$
- (49)  $\prod(c \cdot R) = \prod(i \mapsto c) \cdot \prod R.$

## 5. MODIFIED PART OF [1]

We now state several propositions:

- (50) For every complex-valued finite sequence  $x$  holds  $\text{len}(-x) = \text{len } x.$
- (51) For all complex-valued finite sequences  $x_1, x_2$  such that  $\text{len } x_1 = \text{len } x_2$  holds  $\text{len}(x_1 + x_2) = \text{len } x_1.$
- (52) For all complex-valued finite sequences  $x_1, x_2$  such that  $\text{len } x_1 = \text{len } x_2$  holds  $\text{len}(x_1 - x_2) = \text{len } x_1.$
- (53) For every real number  $a$  and for every complex-valued finite sequence  $x$  holds  $\text{len}(a \cdot x) = \text{len } x.$
- (54) For all complex-valued finite sequences  $x, y, z$  such that  $\text{len } x = \text{len } y = \text{len } z$  holds  $(x + y) \bullet z = x \bullet z + y \bullet z.$

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