

Second-Order Partial Differentiation of Real Ternary Functions

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Summary. In this article, we shall extend the result of [17] to discuss second-order partial differentiation of real ternary functions (refer to [7] and [14] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [6], [11], [12], [1], [2], [3], [4], [5], [7], [16], [17], [13], [8], [15], [10], and [9].

1. SECOND-ORDER PARTIAL DERIVATIVES

For simplicity, we use the following convention: $x, x_0, y, y_0, z, z_0, r$ denote real numbers, u, u_0 denote elements of \mathcal{R}^3 , f, f_1, f_2 denote partial functions from \mathcal{R}^3 to \mathbb{R} , R denotes a rest, and L denotes a linear function.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . We say that f is partial differentiable on 1st-1st coordinate in u if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 1), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 1), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 1st-2nd coordinate in u if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 1), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 1), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 1st-3rd coordinate in u if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 1), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 1), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in u if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 2), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 2), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in u if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 2), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 2), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 2nd-3rd coordinate in u if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 2), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 2), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 3rd-1st coordinate in u if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 3), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 3), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 3), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 3rd-2nd coordinate in u if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 3), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 3), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 3), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 3rd-3rd coordinate in u if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 3), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 3), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 3), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 1st-1st coordinate in u . The functor $\text{hpartdiff11}(f, u)$ yielding a real number is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 1), u)$ and there exist L, R such that $\text{hpartdiff11}(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 1), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 1st-2nd coordinate in u . The functor $\text{hpartdiff12}(f, u)$ yielding a real number is defined by the condition (Def. 11).

(Def. 11) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 1), u)$ and there exist L, R such that $\text{hpartdiff12}(f, u) =$

$L(1)$ and for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 1), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 1st-3rd coordinate in u . The functor $\text{hpartdiff13}(f, u)$ yielding a real number is defined by the condition (Def. 12).

(Def. 12) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 1), u)$ and there exist L, R such that $\text{hpartdiff13}(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 1), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 2nd-1st coordinate in u . The functor $\text{hpartdiff21}(f, u)$ yielding a real number is defined by the condition (Def. 13).

(Def. 13) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 2), u)$ and there exist L, R such that $\text{hpartdiff21}(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 2), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 2nd-2nd coordinate in u . The functor $\text{hpartdiff22}(f, u)$ yielding a real number is defined by the condition (Def. 14).

(Def. 14) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 2), u)$ and there exist L, R such that $\text{hpartdiff22}(f, u) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 2), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 2nd-3rd coordinate in u . The functor $\text{hpartdiff23}(f, u)$ yielding a real number is defined by the condition (Def. 15).

(Def. 15) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 2), u)$ and there exist L, R such that $\text{hpartdiff23}(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 2), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 3rd-1st coordinate in u . The functor

$\text{hpartdiff31}(f, u)$ yields a real number and is defined by the condition (Def. 16).

(Def. 16) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 3), u)$ and there exist L, R such that $\text{hpartdiff31}(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(1, \text{pdiff1}(f, 3), u))(x) - (\text{SVF1}(1, \text{pdiff1}(f, 3), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 3rd-2nd coordinate in u . The functor $\text{hpartdiff32}(f, u)$ yielding a real number is defined by the condition (Def. 17).

(Def. 17) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 3), u)$ and there exist L, R such that $\text{hpartdiff32}(f, u) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF1}(2, \text{pdiff1}(f, 3), u))(y) - (\text{SVF1}(2, \text{pdiff1}(f, 3), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let u be an element of \mathcal{R}^3 . Let us assume that f is partial differentiable on 3rd-3rd coordinate in u . The functor $\text{hpartdiff33}(f, u)$ yielding a real number is defined by the condition (Def. 18).

(Def. 18) There exist real numbers x_0, y_0, z_0 such that

- (i) $u = \langle x_0, y_0, z_0 \rangle$, and
- (ii) there exists a neighbourhood N of z_0 such that $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 3), u)$ and there exist L, R such that $\text{hpartdiff33}(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\text{SVF1}(3, \text{pdiff1}(f, 3), u))(z) - (\text{SVF1}(3, \text{pdiff1}(f, 3), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Next we state a number of propositions:

- (1) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u , then $\text{SVF1}(1, \text{pdiff1}(f, 1), u)$ is differentiable in x_0 .
- (2) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u , then $\text{SVF1}(2, \text{pdiff1}(f, 1), u)$ is differentiable in y_0 .
- (3) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u , then $\text{SVF1}(3, \text{pdiff1}(f, 1), u)$ is differentiable in z_0 .
- (4) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u , then $\text{SVF1}(1, \text{pdiff1}(f, 2), u)$ is differentiable in x_0 .
- (5) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u , then $\text{SVF1}(2, \text{pdiff1}(f, 2), u)$ is differentiable in y_0 .
- (6) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u , then $\text{SVF1}(3, \text{pdiff1}(f, 2), u)$ is differentiable in z_0 .
- (7) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u , then $\text{SVF1}(1, \text{pdiff1}(f, 3), u)$ is differentiable in x_0 .

- (8) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u , then $\text{SVF1}(2, \text{pdiff1}(f, 3), u)$ is differentiable in y_0 .
- (9) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u , then $\text{SVF1}(3, \text{pdiff1}(f, 3), u)$ is differentiable in z_0 .
- (10) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u , then $\text{hpartdiff11}(f, u) = (\text{SVF1}(1, \text{pdiff1}(f, 1), u))'(x_0)$.
- (11) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u , then $\text{hpartdiff12}(f, u) = (\text{SVF1}(2, \text{pdiff1}(f, 1), u))'(y_0)$.
- (12) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u , then $\text{hpartdiff13}(f, u) = (\text{SVF1}(3, \text{pdiff1}(f, 1), u))'(z_0)$.
- (13) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u , then $\text{hpartdiff21}(f, u) = (\text{SVF1}(1, \text{pdiff1}(f, 2), u))'(x_0)$.
- (14) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u , then $\text{hpartdiff22}(f, u) = (\text{SVF1}(2, \text{pdiff1}(f, 2), u))'(y_0)$.
- (15) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u , then $\text{hpartdiff23}(f, u) = (\text{SVF1}(3, \text{pdiff1}(f, 2), u))'(z_0)$.
- (16) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u , then $\text{hpartdiff31}(f, u) = (\text{SVF1}(1, \text{pdiff1}(f, 3), u))'(x_0)$.
- (17) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u , then $\text{hpartdiff32}(f, u) = (\text{SVF1}(2, \text{pdiff1}(f, 3), u))'(y_0)$.
- (18) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u , then $\text{hpartdiff33}(f, u) = (\text{SVF1}(3, \text{pdiff1}(f, 3), u))'(z_0)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. We say that f is partial differentiable on 1st-1st coordinate on D if and only if:

- (Def. 19) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f|_D$ is partial differentiable on 1st-1st coordinate in u .

We say that f is partial differentiable on 1st-2nd coordinate on D if and only if:

- (Def. 20) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f|_D$ is partial differentiable on 1st-2nd coordinate in u .

We say that f is partial differentiable on 1st-3rd coordinate on D if and only if:

- (Def. 21) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f|_D$ is partial differentiable on 1st-3rd coordinate in u .

We say that f is partial differentiable on 2nd-1st coordinate on D if and only if:

- (Def. 22) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f|_D$ is partial differentiable on 2nd-1st coordinate in u .

We say that f is partial differentiable on 2nd-2nd coordinate on D if and only if:

- (Def. 23) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f|_D$ is partial differentiable on 2nd-2nd coordinate in u .

We say that f is partial differentiable on 2nd-3rd coordinate on D if and only if:

(Def. 24) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright D$ is partial differentiable on 2nd-3rd coordinate in u .

We say that f is partial differentiable on 3rd-1st coordinate on D if and only if:

(Def. 25) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright D$ is partial differentiable on 3rd-1st coordinate in u .

We say that f is partial differentiable on 3rd-2nd coordinate on D if and only if:

(Def. 26) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright D$ is partial differentiable on 3rd-2nd coordinate in u .

We say that f is partial differentiable on 3rd-3rd coordinate on D if and only if:

(Def. 27) $D \subseteq \text{dom } f$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright D$ is partial differentiable on 3rd-3rd coordinate in u .

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on D . The functor $f \upharpoonright_D^{1\text{st}-1\text{st}}$ yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined by:

(Def. 28) $\text{dom}(f \upharpoonright_D^{1\text{st}-1\text{st}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright_D^{1\text{st}-1\text{st}}(u) = \text{hpartdiff11}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on D . The functor $f \upharpoonright_D^{1\text{st}-2\text{nd}}$ yielding a partial function from \mathcal{R}^3 to \mathbb{R} is defined by:

(Def. 29) $\text{dom}(f \upharpoonright_D^{1\text{st}-2\text{nd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright_D^{1\text{st}-2\text{nd}}(u) = \text{hpartdiff12}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 1st-3rd coordinate on D . The functor $f \upharpoonright_D^{1\text{st}-3\text{rd}}$ yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined by:

(Def. 30) $\text{dom}(f \upharpoonright_D^{1\text{st}-3\text{rd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright_D^{1\text{st}-3\text{rd}}(u) = \text{hpartdiff13}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on D . The functor $f \upharpoonright_D^{2\text{nd}-1\text{st}}$ yielding a partial function from \mathcal{R}^3 to \mathbb{R} is defined as follows:

(Def. 31) $\text{dom}(f \upharpoonright_D^{2\text{nd}-1\text{st}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright_D^{2\text{nd}-1\text{st}}(u) = \text{hpartdiff21}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on D . The functor $f \upharpoonright_D^{2\text{nd}-2\text{nd}}$ yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined by:

(Def. 32) $\text{dom}(f \upharpoonright_D^{2\text{nd}-2\text{nd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds $f \upharpoonright_D^{2\text{nd}-2\text{nd}}(u) = \text{hpartdiff22}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 2nd-3rd coordinate on D . The functor $f \upharpoonright_D^{2\text{nd}-3\text{rd}}$

yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined by:

- (Def. 33) $\text{dom}(f_{\downarrow D}^{2\text{nd}-3\text{rd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds
 $f_{\downarrow D}^{2\text{nd}-3\text{rd}}(u) = \text{hpartdiff23}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 3rd-1st coordinate on D . The functor $f_{\downarrow D}^{3\text{rd}-1\text{st}}$ yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined as follows:

- (Def. 34) $\text{dom}(f_{\downarrow D}^{3\text{rd}-1\text{st}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds
 $f_{\downarrow D}^{3\text{rd}-1\text{st}}(u) = \text{hpartdiff31}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 3rd-2nd coordinate on D . The functor $f_{\downarrow D}^{3\text{rd}-2\text{nd}}$ yields a partial function from \mathcal{R}^3 to \mathbb{R} and is defined by:

- (Def. 35) $\text{dom}(f_{\downarrow D}^{3\text{rd}-2\text{nd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds
 $f_{\downarrow D}^{3\text{rd}-2\text{nd}}(u) = \text{hpartdiff32}(f, u)$.

Let f be a partial function from \mathcal{R}^3 to \mathbb{R} and let D be a set. Let us assume that f is partial differentiable on 3rd-3rd coordinate on D . The functor $f_{\downarrow D}^{3\text{rd}-3\text{rd}}$ yielding a partial function from \mathcal{R}^3 to \mathbb{R} is defined by:

- (Def. 36) $\text{dom}(f_{\downarrow D}^{3\text{rd}-3\text{rd}}) = D$ and for every element u of \mathcal{R}^3 such that $u \in D$ holds
 $f_{\downarrow D}^{3\text{rd}-3\text{rd}}(u) = \text{hpartdiff33}(f, u)$.

2. MAIN PROPERTIES OF SECOND-ORDER PARTIAL DERIVATIVES

Next we state a number of propositions:

- (19) f is partial differentiable on 1st-1st coordinate in u if and only if $\text{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 1.
- (20) f is partial differentiable on 1st-2nd coordinate in u if and only if $\text{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 2.
- (21) f is partial differentiable on 1st-3rd coordinate in u if and only if $\text{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 3.
- (22) f is partial differentiable on 2nd-1st coordinate in u if and only if $\text{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 1.
- (23) f is partial differentiable on 2nd-2nd coordinate in u if and only if $\text{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 2.
- (24) f is partial differentiable on 2nd-3rd coordinate in u if and only if $\text{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 3.
- (25) f is partial differentiable on 3rd-1st coordinate in u if and only if $\text{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 1.
- (26) f is partial differentiable on 3rd-2nd coordinate in u if and only if $\text{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 2.

- (27) f is partial differentiable on 3rd-3rd coordinate in u if and only if $\text{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 3.
- (28) If f is partial differentiable on 1st-1st coordinate in u , then $\text{hpartdiff11}(f, u) = \text{partdiff}(\text{pdiff1}(f, 1), u, 1)$.
- (29) If f is partial differentiable on 1st-2nd coordinate in u , then $\text{hpartdiff12}(f, u) = \text{partdiff}(\text{pdiff1}(f, 1), u, 2)$.
- (30) If f is partial differentiable on 1st-3rd coordinate in u , then $\text{hpartdiff13}(f, u) = \text{partdiff}(\text{pdiff1}(f, 1), u, 3)$.
- (31) If f is partial differentiable on 2nd-1st coordinate in u , then $\text{hpartdiff21}(f, u) = \text{partdiff}(\text{pdiff1}(f, 2), u, 1)$.
- (32) If f is partial differentiable on 2nd-2nd coordinate in u , then $\text{hpartdiff22}(f, u) = \text{partdiff}(\text{pdiff1}(f, 2), u, 2)$.
- (33) If f is partial differentiable on 2nd-3rd coordinate in u , then $\text{hpartdiff23}(f, u) = \text{partdiff}(\text{pdiff1}(f, 2), u, 3)$.
- (34) If f is partial differentiable on 3rd-1st coordinate in u , then $\text{hpartdiff31}(f, u) = \text{partdiff}(\text{pdiff1}(f, 3), u, 1)$.
- (35) If f is partial differentiable on 3rd-2nd coordinate in u , then $\text{hpartdiff32}(f, u) = \text{partdiff}(\text{pdiff1}(f, 3), u, 2)$.
- (36) If f is partial differentiable on 3rd-3rd coordinate in u , then $\text{hpartdiff33}(f, u) = \text{partdiff}(\text{pdiff1}(f, 3), u, 3)$.
- (37) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(1, 3))(u_0)$. Suppose f is partial differentiable on 1st-1st coordinate in u_0 and $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 1), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(1, 3))(u_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)_*c))$ is convergent and $\text{hpartdiff11}(f, u_0) = \lim(h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)_*c)))$.
- (38) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(2, 3))(u_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in u_0 and $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 1), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2, 3))(u_0)\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)_*c))$ is convergent and $\text{hpartdiff12}(f, u_0) = \lim(h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)_*c)))$.
- (39) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(3, 3))(u_0)$. Suppose f is partial differentiable on 1st-3rd coordinate in u_0 and $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 1), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real num-

bers. Suppose $\text{rng } c = \{(\text{proj}(3,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)_*c))$ is convergent and $\text{hpartdiff13}(f, u_0) = \lim(h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)_*c)))$.

- (40) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in u_0 and $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(1,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\text{hpartdiff21}(f, u_0) = \lim(h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)_*c)))$.
- (41) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in u_0 and $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\text{hpartdiff22}(f, u_0) = \lim(h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)_*c)))$.
- (42) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(3,3))(u_0)$. Suppose f is partial differentiable on 2nd-3rd coordinate in u_0 and $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(3,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\text{hpartdiff23}(f, u_0) = \lim(h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 2), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 2), u_0)_*c)))$.
- (43) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 3rd-1st coordinate in u_0 and $N \subseteq \text{dom SVF1}(1, \text{pdiff1}(f, 3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(1,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 3), u_0)_*c))$ is convergent and $\text{hpartdiff31}(f, u_0) = \lim(h^{-1}((\text{SVF1}(1, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(1, \text{pdiff1}(f, 3), u_0)_*c)))$.
- (44) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 3rd-2nd coordinate in u_0 and $N \subseteq \text{dom SVF1}(2, \text{pdiff1}(f, 3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2,3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then

- $h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 3), u_0)_*c))$ is convergent and $\text{hpartdiff32}(f, u_0) = \lim(h^{-1}((\text{SVF1}(2, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(2, \text{pdiff1}(f, 3), u_0)_*c)))$.
- (45) Let u_0 be an element of \mathcal{R}^3 and N be a neighbourhood of $(\text{proj}(3, 3))(u_0)$. Suppose f is partial differentiable on 3rd-3rd coordinate in u_0 and $N \subseteq \text{dom SVF1}(3, \text{pdiff1}(f, 3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(3, 3))(u_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 3), u_0)_*c))$ is convergent and $\text{hpartdiff33}(f, u_0) = \lim(h^{-1}((\text{SVF1}(3, \text{pdiff1}(f, 3), u_0)_*(h+c)) - (\text{SVF1}(3, \text{pdiff1}(f, 3), u_0)_*c)))$.
- (46) Suppose that
- (i) f_1 is partial differentiable on 1st-1st coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 1 and $\text{partdiff}(\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1), u_0, 1) = \text{hpartdiff11}(f_1, u_0) + \text{hpartdiff11}(f_2, u_0)$.
- (47) Suppose that
- (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 2 and $\text{partdiff}(\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1), u_0, 2) = \text{hpartdiff12}(f_1, u_0) + \text{hpartdiff12}(f_2, u_0)$.
- (48) Suppose that
- (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 3 and $\text{partdiff}(\text{pdiff1}(f_1, 1) + \text{pdiff1}(f_2, 1), u_0, 3) = \text{hpartdiff13}(f_1, u_0) + \text{hpartdiff13}(f_2, u_0)$.
- (49) Suppose that
- (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1 and $\text{partdiff}(\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2), u_0, 1) = \text{hpartdiff21}(f_1, u_0) + \text{hpartdiff21}(f_2, u_0)$.
- (50) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2 and $\text{partdiff}(\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2), u_0, 2) = \text{hpartdiff22}(f_1, u_0) + \text{hpartdiff22}(f_2, u_0)$.

(51) Suppose that

- (i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0 .

Then $\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and $\text{partdiff}(\text{pdiff1}(f_1, 2) + \text{pdiff1}(f_2, 2), u_0, 3) = \text{hpartdiff23}(f_1, u_0) + \text{hpartdiff23}(f_2, u_0)$.

(52) Suppose that

- (i) f_1 is partial differentiable on 1st-1st coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0 .

Then $\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 1 and $\text{partdiff}(\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1), u_0, 1) = \text{hpartdiff11}(f_1, u_0) - \text{hpartdiff11}(f_2, u_0)$.

(53) Suppose that

- (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0 .

Then $\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 2 and $\text{partdiff}(\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1), u_0, 2) = \text{hpartdiff12}(f_1, u_0) - \text{hpartdiff12}(f_2, u_0)$.

(54) Suppose that

- (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0 .

Then $\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 3 and $\text{partdiff}(\text{pdiff1}(f_1, 1) - \text{pdiff1}(f_2, 1), u_0, 3) = \text{hpartdiff13}(f_1, u_0) - \text{hpartdiff13}(f_2, u_0)$.

(55) Suppose that

- (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0 .

Then $\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1 and $\text{partdiff}(\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2), u_0, 1) = \text{hpartdiff21}(f_1, u_0) - \text{hpartdiff21}(f_2, u_0)$.

(56) Suppose that

- (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0 .

Then $\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2 and $\text{partdiff}(\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2), u_0, 2) = \text{hpartdiff22}(f_1, u_0) - \text{hpartdiff22}(f_2, u_0)$.

(57) Suppose that

- (i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0 , and
- (ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0 .

Then $\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and $\text{partdiff}(\text{pdiff1}(f_1, 2) - \text{pdiff1}(f_2, 2), u_0, 3) = \text{hpartdiff23}(f_1, u_0) - \text{hpartdiff23}(f_2, u_0)$.

- $\text{hpartdiff23}(f_2, u_0)$.
- (58) Suppose f is partial differentiable on 1st-1st coordinate in u_0 .
Then $r \text{ pdiff1}(f, 1)$ is partially differentiable in u_0 w.r.t. 1 and
 $\text{partdiff}(r \text{ pdiff1}(f, 1), u_0, 1) = r \cdot \text{hpartdiff11}(f, u_0)$.
- (59) Suppose f is partial differentiable on 1st-2nd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 1)$ is partially differentiable in u_0 w.r.t. 2 and
 $\text{partdiff}(r \text{ pdiff1}(f, 1), u_0, 2) = r \cdot \text{hpartdiff12}(f, u_0)$.
- (60) Suppose f is partial differentiable on 1st-3rd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 1)$ is partially differentiable in u_0 w.r.t. 3 and
 $\text{partdiff}(r \text{ pdiff1}(f, 1), u_0, 3) = r \cdot \text{hpartdiff13}(f, u_0)$.
- (61) Suppose f is partial differentiable on 2nd-1st coordinate in u_0 .
Then $r \text{ pdiff1}(f, 2)$ is partially differentiable in u_0 w.r.t. 1 and
 $\text{partdiff}(r \text{ pdiff1}(f, 2), u_0, 1) = r \cdot \text{hpartdiff21}(f, u_0)$.
- (62) Suppose f is partial differentiable on 2nd-2nd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 2)$ is partially differentiable in u_0 w.r.t. 2 and
 $\text{partdiff}(r \text{ pdiff1}(f, 2), u_0, 2) = r \cdot \text{hpartdiff22}(f, u_0)$.
- (63) Suppose f is partial differentiable on 2nd-3rd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 2)$ is partially differentiable in u_0 w.r.t. 3 and
 $\text{partdiff}(r \text{ pdiff1}(f, 2), u_0, 3) = r \cdot \text{hpartdiff23}(f, u_0)$.
- (64) Suppose f is partial differentiable on 3rd-1st coordinate in u_0 .
Then $r \text{ pdiff1}(f, 3)$ is partially differentiable in u_0 w.r.t. 1 and
 $\text{partdiff}(r \text{ pdiff1}(f, 3), u_0, 1) = r \cdot \text{hpartdiff31}(f, u_0)$.
- (65) Suppose f is partial differentiable on 3rd-2nd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 3)$ is partially differentiable in u_0 w.r.t. 2 and
 $\text{partdiff}(r \text{ pdiff1}(f, 3), u_0, 2) = r \cdot \text{hpartdiff32}(f, u_0)$.
- (66) Suppose f is partial differentiable on 3rd-3rd coordinate in u_0 .
Then $r \text{ pdiff1}(f, 3)$ is partially differentiable in u_0 w.r.t. 3 and
 $\text{partdiff}(r \text{ pdiff1}(f, 3), u_0, 3) = r \cdot \text{hpartdiff33}(f, u_0)$.
- (67) Suppose that
- (i) f_1 is partial differentiable on 1st-1st coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) \text{ pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 1.
- (68) Suppose that
- (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) \text{ pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 2.
- (69) Suppose that
- (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 1) \text{ pdiff1}(f_2, 1)$ is partially differentiable in u_0 w.r.t. 3.

- (70) Suppose that
- (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 2) \text{ pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1.
- (71) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 2) \text{ pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2.
- (72) Suppose that
- (i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 2) \text{ pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3.
- (73) Suppose that
- (i) f_1 is partial differentiable on 3rd-1st coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 3rd-1st coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 3) \text{ pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 1.
- (74) Suppose that
- (i) f_1 is partial differentiable on 3rd-2nd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 3rd-2nd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 3) \text{ pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 2.
- (75) Suppose that
- (i) f_1 is partial differentiable on 3rd-3rd coordinate in u_0 , and
 - (ii) f_2 is partial differentiable on 3rd-3rd coordinate in u_0 .
- Then $\text{pdiff1}(f_1, 3) \text{ pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 3.
- (76) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 1st-1st coordinate in u_0 . Then $\text{SVF1}(1, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(1, 3))(u_0)$.
- (77) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 1st-2nd coordinate in u_0 . Then $\text{SVF1}(2, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(2, 3))(u_0)$.
- (78) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 1st-3rd coordinate in u_0 . Then $\text{SVF1}(3, \text{pdiff1}(f, 1), u_0)$ is continuous in $(\text{proj}(3, 3))(u_0)$.
- (79) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 2nd-1st coordinate in u_0 . Then $\text{SVF1}(1, \text{pdiff1}(f, 2), u_0)$ is continuous in $(\text{proj}(1, 3))(u_0)$.
- (80) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 2nd-2nd coordinate in u_0 . Then $\text{SVF1}(2, \text{pdiff1}(f, 2), u_0)$ is continuous in $(\text{proj}(2, 3))(u_0)$.

- (81) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 2nd-3rd coordinate in u_0 . Then $\text{SVF1}(3, \text{pdiff1}(f, 2), u_0)$ is continuous in $(\text{proj}(3, 3))(u_0)$.
- (82) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 3rd-1st coordinate in u_0 . Then $\text{SVF1}(1, \text{pdiff1}(f, 3), u_0)$ is continuous in $(\text{proj}(1, 3))(u_0)$.
- (83) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 3rd-2nd coordinate in u_0 . Then $\text{SVF1}(2, \text{pdiff1}(f, 3), u_0)$ is continuous in $(\text{proj}(2, 3))(u_0)$.
- (84) Let u_0 be an element of \mathcal{R}^3 . Suppose f is partial differentiable on 3rd-3rd coordinate in u_0 . Then $\text{SVF1}(3, \text{pdiff1}(f, 3), u_0)$ is continuous in $(\text{proj}(3, 3))(u_0)$.

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