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Integrability Formulas. Part III

Bo Li
Qingdao University of Science and Technology
China

Na Ma
Qingdao University of Science and Technology
China

Summary. In this article, we give several differentiation and integrability formulas of composite trigonometric function.

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The papers [9], [10], [15], [2], [3], [1], [6], [11], [4], [16], [7], [8], [5], [17], [13], [14], and [12] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following convention: a, x denote real numbers, n denotes a natural number, A denotes a closed-interval subset of $\mathbb{R}$, f, f_1 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1) Suppose $Z \subseteq \mathrm{dom}((\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})$. Then

(i) $-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot \frac{1}{\mathrm{id}_Z})'_{\restriction Z}(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^{\mathbf{2}} \cdot (\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function exp}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function exp})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$.

(3) Suppose $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function ln}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function ln})$ is differentiable on Z, and

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(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function ln}))'_{\restriction Z}(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function exp}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function exp}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = \frac{(\text{the function exp})((\text{the function cosec})(x)) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$. Then

(i) $-(\text{the function ln}) \cdot (\text{the function cosec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function ln}) \cdot (\text{the function cosec}))'_{\restriction Z}(x) = (\text{the function cot})(x)$.

(6) Suppose $Z \subseteq \text{dom}((\square^n) \cdot \text{the function cosec})$ and $1 \leq n$. Then

(i) $-(\square^n) \cdot \text{the function cosec}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\square^n) \cdot \text{the function cosec})'_{\restriction Z}(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$.

(7) Suppose $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function sec})$. Then

(i) $-\frac{1}{\text{id}_Z}$ the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\text{id}_Z} \text{ the function sec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x^{\mathbf{2}}} - \frac{\frac{(\text{the function sin})(x)}{x}}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(8) Suppose $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function cosec})$. Then

(i) $-\frac{1}{\text{id}_Z}$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\frac{1}{\text{id}_Z} \text{ the function cosec})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x^{\mathbf{2}}} + \frac{\frac{(\text{the function cos})(x)}{x}}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(9) Suppose $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function sin}))'_{\restriction Z}(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{\mathbf{2}}}$.

(10) Suppose $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function cot}))$. Then

(i) $-(\text{the function sec}) \cdot (\text{the function cot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function sec}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}}$.

(11) Suppose $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$. Then

(i) $-(\text{the function cosec}) \cdot (\text{the function tan})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cosec}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = \frac{\frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}}$.

(12) Suppose $Z \subseteq \text{dom}((\text{the function cot})\ (\text{the function sec}))$. Then

(i) $-(\text{the function cot})\,(\text{the function sec})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-(\text{the function cot})\,(\text{the function$

sec))$'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^2}}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x)\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.

(13) Suppose $Z \subseteq$ dom((the function cot) (the function cosec)). Then

(i) $-$(the function cot) (the function cosec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$(the function cot) (the function cosec))$'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function sin})(x)^2}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.

(14) Suppose $Z \subseteq$ dom((the function cos) (the function cot)). Then

(i) $-$(the function cos) (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$(the function cos) (the function cot))$'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.

2. Integrability Formulas

We now state a number of propositions:

(15) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^2\cdot(\text{the function cos})(\frac{1}{x})^2}$,

(iii) $Z \subseteq$ dom((the function sec) $\cdot \frac{1}{\mathrm{id}_Z}$),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function sec})\cdot\frac{1}{\mathrm{id}_Z})(\sup A) - (-(\text{the function sec})\cdot\frac{1}{\mathrm{id}_Z})(\inf A)$.

(16) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(\frac{1}{x})}{x^2\cdot(\text{the function sin})(\frac{1}{x})^2}$,

(iii) $Z \subseteq$ dom((the function cosec) $\cdot \frac{1}{\mathrm{id}_Z}$),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function cosec})\cdot\frac{1}{\mathrm{id}_Z})(\sup A) - ((\text{the function cosec})\cdot\frac{1}{\mathrm{id}_Z})(\inf A)$.

(17) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)\cdot(\text{the function sin})((\text{the function exp})(x))}{(\text{the function cos})((\text{the function exp})(x))^2}$,

(iii) $Z \subseteq$ dom((the function sec) $\cdot$(the function exp)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sec) $\cdot$(the function exp))(sup A)$-$((the function sec) $\cdot$(the function exp))(inf A).

(18) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function exp)}(x)\cdot\text{(the function cos)}(\text{(the function exp)}(x))}{\text{(the function sin)}(\text{(the function exp)}(x))^{\mathbf{2}}}$,
- (iii) $Z \subseteq$ dom((the function cosec) $\cdot$(the function exp)),
- (iv) $Z = \operatorname{dom} f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx$ = ($-$(the function cosec) $\cdot$ (the function exp))(sup A) $-$ ($-$(the function cosec) $\cdot$ (the function exp))(inf A).

(19) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function sin)}(\text{(the function ln)}(x))}{x\cdot\text{(the function cos)}(\text{(the function ln)}(x))^{\mathbf{2}}}$,
- (iii) $Z \subseteq$ dom((the function sec) $\cdot$(the function ln)),
- (iv) $Z = \operatorname{dom} f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sec) $\cdot$(the function ln))(sup A) $-$ ((the function sec) $\cdot$(the function ln))(inf A).

(20) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function cos)}(\text{(the function ln)}(x))}{x\cdot\text{(the function sin)}(\text{(the function ln)}(x))^{\mathbf{2}}}$,
- (iii) $Z \subseteq$ dom((the function cosec) $\cdot$(the function ln)),
- (iv) $Z = \operatorname{dom} f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ($-$(the function cosec) $\cdot$ (the function ln))(sup A) $-$ ($-$(the function cosec) $\cdot$ (the function ln))(inf A).

(21) Suppose that
- (i) $A \subseteq Z$,
- (ii) $f =$ ((the function exp) $\cdot$(the function sec)) $\frac{\text{the function sin}}{\text{(the function cos)}^{\mathbf{2}}}$,
- (iii) $Z = \operatorname{dom} f$, and
- (iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (($the function exp$) \cdot ($the function sec$))(\sup A) - (($the function exp$) \cdot ($the function sec$))(\inf A)$.

(22) Suppose that

(i) $A \subseteq Z$,

(ii) $f = (($the function exp$) \cdot ($the function cosec$))\ \frac{\text{the function cos}}{(\text{the function sin})^{\mathbf{2}}}$,

(iii) $Z = \operatorname{dom} f$, and

(iv) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-($the function exp$) \cdot ($the function cosec$))(\sup A) - (-($the function exp$) \cdot ($the function cosec$))(\inf A)$.

(23) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}(($the function ln$) \cdot ($the function sec$))$,

(iii) $Z = \operatorname{dom}\,($the function tan$)$, and

(iv) (the function tan)$\restriction A$ is continuous.

Then $\int\limits_A ($the function tan$)(x)dx = (($the function ln$) \cdot ($the function sec$))(\sup A) - (($the function ln$) \cdot ($the function sec$))(\inf A)$.

(24) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}(($the function ln$) \cdot ($the function cosec$))$,

(iii) $Z = \operatorname{dom}\,($the function cot$)$, and

(iv) ($-$the function cot)$\restriction A$ is continuous.

Then $\int\limits_A (-$the function cot$)(x)dx = (($the function ln$) \cdot ($the function cosec$))(\sup A) - (($the function ln$) \cdot ($the function cosec$))(\inf A)$.

(25) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}(($the function ln$) \cdot ($the function cosec$))$,

(iii) $Z = \operatorname{dom}\,($the function cot$)$, and

(iv) (the function cot)$\restriction A$ is continuous.

Then $\int\limits_A ($the function cot$)(x)dx = (-($the function ln$) \cdot ($the function cosec$))(\sup A) - (-($the function ln$) \cdot ($the function cosec$))(\inf A)$.

(26) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$,

(iii) $Z \subseteq \operatorname{dom}((\square^n) \cdot$ the function sec$)$,

(iv) $1 \leq n$,

(v) $Z = \mathrm{dom}\, f$, and
(vi) $f{\restriction}A$ is continuous.
Then $\int\limits_A f(x)dx = ((\square^n) \cdot$ the function sec$)(\sup A) - ((\square^n) \cdot$ the function sec$)(\inf A)$.

(27) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$,
(iii) $Z \subseteq \mathrm{dom}((\square^n) \cdot$ the function cosec$)$,
(iv) $1 \leq n$,
(v) $Z = \mathrm{dom}\, f$, and
(vi) $f{\restriction}A$ is continuous.
Then $\int\limits_A f(x)dx = (-(\square^n) \cdot$ the function cosec$)(\sup A) - (-(\square^n) \cdot$ the function cosec$)(\inf A)$.

(28) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}(($the function exp$)$ $($the function sec$))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.
Then $\int\limits_A f(x)dx = (($the function exp$)$ $($the function sec$))(\sup A) - (($the function exp$)$ $($the function sec$))(\inf A)$.

(29) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}(($the function exp$)$ $($the function cosec$))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.
Then $\int\limits_A f(x)dx = (($the function exp$)$ $($the function cosec$))(\sup A) - (($the function exp$)$ $($the function cosec$))(\inf A)$.

(30) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function sin})(a \cdot x) - (\text{the function cos})(a \cdot x)^{\mathbf{2}}}{(\text{the function cos})(a \cdot x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - (\frac{1}{a}$ ((the function sec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(31) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\text{(the function cos)}(a \cdot x) - \text{(the function sin)}(a \cdot x)^2}{\text{(the function sin)}(a \cdot x)^2}$,
(iii) $Z \subseteq \operatorname{dom}((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\sup A) - ((-\frac{1}{a})$ ((the function cosec) $\cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(32) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{\text{(the function cos)}(x)}}{x} + \frac{\text{(the function ln)}(x) \cdot \text{(the function sin)}(x)}{\text{(the function cos)}(x)^2}$,
(iii) $Z \subseteq \operatorname{dom}$((the function ln) (the function sec)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function sec))$(\sup A) -$ ((the function ln) (the function sec))$(\inf A)$.

(33) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{\text{(the function sin)}(x)}}{x} - \frac{\text{(the function ln)}(x) \cdot \text{(the function cos)}(x)}{\text{(the function sin)}(x)^2}$,
(iii) $Z \subseteq \operatorname{dom}$((the function ln) (the function cosec)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function ln) (the function cosec))$(\sup A) -$ ((the function ln) (the function cosec))$(\inf A)$.

(34) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)}}{x^{\mathbf{2}}} - \frac{\frac{(\text{the function sin})(x)}{x}}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function sec),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function sec)(sup A) $-$ $(-\frac{1}{\mathrm{id}_Z}$ the function sec)(inf A).

(35) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)}}{x^{\mathbf{2}}} + \frac{\frac{(\text{the function cos})(x)}{x}}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}(\frac{1}{\mathrm{id}_Z}$ the function cosec),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\frac{1}{\mathrm{id}_Z}$ the function cosec)(sup A) $-$ $(-\frac{1}{\mathrm{id}_Z}$ the function cosec)(inf A).

(36) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(x)\cdot(\text{the function sin})((\text{the function sin})(x))}{(\text{the function cos})((\text{the function sin})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}$((the function sec) $\cdot$(the function sin)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx =$ ((the function sec) $\cdot$(the function sin))(sup A)$-$((the function sec) $\cdot$(the function sin))(inf A).

(37) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(x)\cdot(\text{the function sin})((\text{the function cos})(x))}{(\text{the function cos})((\text{the function cos})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \operatorname{dom}$((the function sec) $\cdot$(the function cos)),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx =$ ($-$(the function sec) $\cdot$ (the function cos))(sup A) $-$ ($-$(the function sec) $\cdot$ (the function cos))(inf A).

(38) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^{2}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function sin}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function sin}))(\inf A)$.

(39) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{(\text{the function sin})(x) \cdot (\text{the function cos})((\text{the function cos})(x))}{(\text{the function sin})((\text{the function cos})(x))^{2}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function cos}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cos}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cos}))(\inf A)$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{\frac{(\text{the function sin})((\text{the function tan})(x))}{(\text{the function cos})(x)^{2}}}{(\text{the function cos})((\text{the function tan})(x))^{2}}$,

(iii) $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function tan}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function sec}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function tan}))(\inf A)$.

(41) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{\frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^{2}}}{(\text{the function cos})((\text{the function cot})(x))^{2}}$,

(iii) $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function cot}))$,

(iv) $Z = \text{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function sec}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function sec}) \cdot (\text{the function cot}))(\inf A)$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds

$f(x) = \frac{\frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function tan}))(\inf A)$.

(43) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds

$f(x) = \frac{\frac{(\text{the function cos})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cosec}) \cdot (\text{the function cot}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function cosec}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cosec}) \cdot (\text{the function cot}))(\inf A)$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function cos})(x)} + \frac{(\text{the function tan})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan})\ (\text{the function sec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\text{the function tan})\ (\text{the function sec}))(\sup A) - ((\text{the function tan})\ (\text{the function sec}))(\inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function sec}))$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-(\text{the function cot})\ (\text{the function sec}))(\sup A) - (-(\text{the}$

function cot) (the function sec))(inf A).

(46) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}}{(\text{the function sin})(x)} - \frac{(\text{the function tan})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan})\ (\text{the function cosec}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = ((\text{the function tan})\ (\text{the function cosec}))(\sup A) - ((\text{the function tan})\ (\text{the function cosec}))(\inf A)$.

(47) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x)\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function cot})\ (\text{the function cosec}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function cot})\,(\text{the function cosec}))(\sup A) - (-(\text{the function cot})\,(\text{the function cosec}))(\inf A)$.

(48) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{1}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan})\ \cdot(\text{the function cot}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-(\text{the function tan}) \cdot (\text{the function cot}))(\sup A) - (-(\text{the function tan}) \cdot (\text{the function cot}))(\inf A)$.

(49) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds
$f(x) = \frac{1}{(\text{the function cos})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$,

(iii) $Z \subseteq \text{dom}((\text{the function tan})\ \cdot(\text{the function tan}))$,

(iv) $Z = \text{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function tan}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function tan}) \cdot (\text{the function tan}))(\inf A)$.

(50) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq \mathrm{dom}((\text{the function cot}) \cdot (\text{the function cot}))$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function cot}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cot}) \cdot (\text{the function cot}))(\inf A)$.

(51) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function sin})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq \mathrm{dom}((\text{the function cot}) \cdot (\text{the function tan}))$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = (-(\text{the function cot}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cot}) \cdot (\text{the function tan}))(\inf A)$.

(52) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} + \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) - (\text{the function cot}))$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int_A f(x)dx = ((\text{the function tan}) - (\text{the function cot}))(\sup A) - ((\text{the function tan}) - (\text{the function cot}))(\inf A)$.

(53) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} - \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) + (\text{the function cot}))$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function tan)+(the function cot))(sup A) − ((the function tan)+(the function cot))(inf A).

(54) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function sin)(x)) · (the function cos)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sin) ·(the function sin))(sup A)−((the function sin) ·(the function sin))(inf A).

(55) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)((the function cos)(x)) · (the function sin)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ (−(the function sin) · (the function cos))(sup A) − (−(the function sin) · (the function cos))(inf A).

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function sin)(x)) · (the function cos)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ (−(the function cos) · (the function sin))(sup A) − (−(the function cos) · (the function sin))(inf A).

(57) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)((the function cos)(x)) · (the function sin)(x),

(iii) $Z = \operatorname{dom} f$, and

(iv) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function cos) ·(the function cos))(sup A)−((the function cos) ·(the function cos))(inf A).

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function cos)(x) + $\frac{\text{(the function cos)}(x)}{\text{(the function sin)}(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq$ dom((the function cos) (the function cot)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ (−(the function cos) (the function cot))(sup A) − (−(the function cos) (the function cot))(inf A).

(59) Suppose that

(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) =$ (the function sin)(x) + $\frac{\text{(the function sin)}(x)}{\text{(the function cos)}(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq$ dom((the function sin) (the function tan)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx =$ ((the function sin) (the function tan))(sup A)−((the function sin) (the function tan))(inf A).

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