

## Integrability Formulas. Part III

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**Summary.** In this article, we give several differentiation and integrability formulas of composite trigonometric function.

MML identifier: INTEGR14, version: 7.11.07 4.156.1112

The papers [9], [10], [15], [2], [3], [1], [6], [11], [4], [16], [7], [8], [5], [17], [13], [14], and [12] provide the terminology and notation for this paper.

### 1. DIFFERENTIATION FORMULAS

For simplicity, we adopt the following convention:  $a, x$  denote real numbers,  $n$  denotes a natural number,  $A$  denotes a closed-interval subset of  $\mathbb{R}$ ,  $f, f_1$  denote partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ , and  $Z$  denotes an open subset of  $\mathbb{R}$ .

One can prove the following propositions:

- (1) Suppose  $Z \subseteq \text{dom}((\text{the function sec}) \cdot \frac{1}{\text{id}_Z})$ . Then
  - (i)  $-(\text{the function sec}) \cdot \frac{1}{\text{id}_Z}$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function sec}) \cdot \frac{1}{\text{id}_Z})'_{|Z}(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^2 \cdot (\text{the function cos})(\frac{1}{x})^2}$ .
- (2) Suppose  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function exp}))$ . Then
  - (i)  $-(\text{the function cosec}) \cdot (\text{the function exp})$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function cosec}) \cdot (\text{the function exp}))'_{|Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(\text{the function exp}(x))}{(\text{the function sin})(\text{the function exp}(x))^2}$ .
- (3) Suppose  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function ln}))$ . Then
  - (i)  $-(\text{the function cosec}) \cdot (\text{the function ln})$  is differentiable on  $Z$ , and

- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function cosec}) \cdot (\text{the function ln}))'_{|Z}(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^2}$ .
- (4) Suppose  $Z \subseteq \text{dom}((\text{the function exp}) \cdot (\text{the function cosec}))$ . Then
- (i)  $-(\text{the function exp}) \cdot (\text{the function cosec})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function exp}) \cdot (\text{the function cosec}))'_{|Z}(x) = \frac{(\text{the function exp})((\text{the function cosec})(x)) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ .
- (5) Suppose  $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$ . Then
- (i)  $-(\text{the function ln}) \cdot (\text{the function cosec})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function ln}) \cdot (\text{the function cosec}))'_{|Z}(x) = (\text{the function cot})(x)$ .
- (6) Suppose  $Z \subseteq \text{dom}((\square^n) \cdot \text{the function cosec})$  and  $1 \leq n$ . Then
- (i)  $-(\square^n) \cdot \text{the function cosec}$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\square^n) \cdot \text{the function cosec})'_{|Z}(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$ .
- (7) Suppose  $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{the function sec})$ . Then
- (i)  $-\frac{1}{\text{id}_Z} \text{the function sec}$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-\frac{1}{\text{id}_Z} \text{the function sec})'_{|Z}(x) = \frac{1}{(\text{the function cos})(x)^2} - \frac{(\text{the function sin})(x)}{(\text{the function cos})(x)^2}$ .
- (8) Suppose  $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{the function cosec})$ . Then
- (i)  $-\frac{1}{\text{id}_Z} \text{the function cosec}$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-\frac{1}{\text{id}_Z} \text{the function cosec})'_{|Z}(x) = \frac{1}{(\text{the function sin})(x)^2} + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ .
- (9) Suppose  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function sin}))$ . Then
- (i)  $-(\text{the function cosec}) \cdot (\text{the function sin})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function cosec}) \cdot (\text{the function sin}))'_{|Z}(x) = \frac{(\text{the function cos})(x) \cdot (\text{the function cos})((\text{the function sin})(x))}{(\text{the function sin})((\text{the function sin})(x))^2}$ .
- (10) Suppose  $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function cot}))$ . Then
- (i)  $-(\text{the function sec}) \cdot (\text{the function cot})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function sec}) \cdot (\text{the function cot}))'_{|Z}(x) = \frac{(\text{the function sin})((\text{the function cot})(x))}{(\text{the function sin})(x)^2} \cdot (\text{the function cos})((\text{the function cot})(x))$ .
- (11) Suppose  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function tan}))$ . Then
- (i)  $-(\text{the function cosec}) \cdot (\text{the function tan})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function cosec}) \cdot (\text{the function tan}))'_{|Z}(x) = \frac{(\text{the function cos})((\text{the function tan})(x))}{(\text{the function cos})(x)^2} \cdot (\text{the function sin})((\text{the function tan})(x))$ .
- (12) Suppose  $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function sec}))$ . Then
- (i)  $-(\text{the function cot}) \cdot (\text{the function sec})$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(-(\text{the function cot}) \cdot (\text{the function sec}))'_{|Z}(x) = \frac{(\text{the function sin})((\text{the function sec})(x))}{(\text{the function sin})(x)^2} \cdot (\text{the function cos})((\text{the function sec})(x))$ .

$$\text{sec})'_{|Z}(x) = \frac{1}{(\text{the function cos})(x)} - \frac{(\text{the function cot})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}.$$

- (13) Suppose  $Z \subseteq \text{dom}((\text{the function cot}) (\text{the function cosec}))$ . Then
- (i)  $-(\text{the function cot}) (\text{the function cosec})$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $-(\text{the function cot}) (\text{the function$

$$\text{cosec})'_{|Z}(x) = \frac{1}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}.$$

- (14) Suppose  $Z \subseteq \text{dom}((\text{the function cos}) (\text{the function cot}))$ . Then
- (i)  $-(\text{the function cos}) (\text{the function cot})$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $-(\text{the function cos}) (\text{the function$

$$\text{cot})'_{|Z}(x) = (\text{the function cos})(x) + \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^2}.$$

## 2. INTEGRABILITY FORMULAS

We now state a number of propositions:

- (15) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function sin})(\frac{1}{x})}{x^2 \cdot (\text{the function cos})(\frac{1}{x})^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function sec}) \cdot \frac{1}{\text{id}_Z})$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = (-(\text{the function sec}) \cdot \frac{1}{\text{id}_Z})(\text{sup } A) - (-(\text{the function sec}) \cdot \frac{1}{\text{id}_Z})(\text{inf } A).$$

- (16) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function cos})(\frac{1}{x})}{x^2 \cdot (\text{the function sin})(\frac{1}{x})^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot \frac{1}{\text{id}_Z})$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((\text{the function cosec}) \cdot \frac{1}{\text{id}_Z})(\text{sup } A) - ((\text{the function cosec}) \cdot \frac{1}{\text{id}_Z})(\text{inf } A).$$

- (17) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})((\text{the function exp})(x))}{(\text{the function cos})((\text{the function exp})(x))^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function exp}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function sec}) \cdot (\text{the function exp}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function exp}))(\inf A)$ .

(18) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^2},$$

(iii)  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function exp}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function exp}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function exp}))(\inf A)$ .

(19) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{(\text{the function sin})((\text{the function ln})(x))}{x \cdot (\text{the function cos})((\text{the function ln})(x))^2},$$

(iii)  $Z \subseteq \text{dom}((\text{the function sec}) \cdot (\text{the function ln}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function sec}) \cdot (\text{the function ln}))(\sup A) - ((\text{the function sec}) \cdot (\text{the function ln}))(\inf A)$ .

(20) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^2},$$

(iii)  $Z \subseteq \text{dom}((\text{the function cosec}) \cdot (\text{the function ln}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-(\text{the function cosec}) \cdot (\text{the function ln}))(\sup A) - (-(\text{the function cosec}) \cdot (\text{the function ln}))(\inf A)$ .

(21) Suppose that

(i)  $A \subseteq Z$ ,

(ii)  $f = ((\text{the function exp}) \cdot (\text{the function sec})) \frac{\text{the function sin}}{(\text{the function cos})^2}$ ,

(iii)  $Z = \text{dom } f$ , and

(iv)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function exp}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function exp}) \cdot (\text{the function sec}))(\inf A)$ .

(22) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii)  $f = ((\text{the function exp}) \cdot (\text{the function cosec})) \frac{\text{the function cos}}{(\text{the function sin})^2}$ ,
- (iii)  $Z = \text{dom } f$ , and
- (iv)  $f|_A$  is continuous.

Then  $\int_A f(x)dx = (-(\text{the function exp}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function exp}) \cdot (\text{the function cosec}))(\inf A)$ .

(23) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii)  $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function sec}))$ ,
- (iii)  $Z = \text{dom}(\text{the function tan})$ , and
- (iv)  $(\text{the function tan})|_A$  is continuous.

Then  $\int_A (\text{the function tan})(x)dx = ((\text{the function ln}) \cdot (\text{the function sec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function sec}))(\inf A)$ .

(24) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii)  $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$ ,
- (iii)  $Z = \text{dom}(\text{the function cot})$ , and
- (iv)  $(-\text{the function cot})|_A$  is continuous.

Then  $\int_A (-\text{the function cot})(x)dx = ((\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$ .

(25) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii)  $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function cosec}))$ ,
- (iii)  $Z = \text{dom}(\text{the function cot})$ , and
- (iv)  $(\text{the function cot})|_A$  is continuous.

Then  $\int_A (\text{the function cot})(x)dx = (-(\text{the function ln}) \cdot (\text{the function cosec}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function cosec}))(\inf A)$ .

(26) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{n \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$ ,
- (iii)  $Z \subseteq \text{dom}((\square^n) \cdot \text{the function sec})$ ,
- (iv)  $1 \leq n$ ,

- (v)  $Z = \text{dom } f$ , and  
 (vi)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\square^n) \cdot \text{the function sec})(\text{sup } A) - ((\square^n) \cdot \text{the function sec})(\text{inf } A)$ .

(27) Suppose that

- (i)  $A \subseteq Z$ ,  
 (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$ ,  
 (iii)  $Z \subseteq \text{dom}((\square^n) \cdot \text{the function cosec})$ ,  
 (iv)  $1 \leq n$ ,  
 (v)  $Z = \text{dom } f$ , and  
 (vi)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\square^n) \cdot \text{the function cosec}(\text{sup } A) - (-\square^n) \cdot \text{the function cosec}(\text{inf } A)$ .

(28) Suppose that

- (i)  $A \subseteq Z$ ,  
 (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$ ,  
 (iii)  $Z \subseteq \text{dom}((\text{the function exp}) (\text{the function sec}))$ ,  
 (iv)  $Z = \text{dom } f$ , and  
 (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function exp}) (\text{the function sec}))(\text{sup } A) - ((\text{the function exp}) (\text{the function sec}))(\text{inf } A)$ .

(29) Suppose that

- (i)  $A \subseteq Z$ ,  
 (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ ,  
 (iii)  $Z \subseteq \text{dom}((\text{the function exp}) (\text{the function cosec}))$ ,  
 (iv)  $Z = \text{dom } f$ , and  
 (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function exp}) (\text{the function cosec}))(\text{sup } A) - ((\text{the function exp}) (\text{the function cosec}))(\text{inf } A)$ .

(30) Suppose that

- (i)  $A \subseteq Z$ ,  
 (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function sin})(a \cdot x) - (\text{the function cos})(a \cdot x)^2}{(\text{the function cos})(a \cdot x)^2}$ ,

- (iii)  $Z \subseteq \text{dom}(\frac{1}{a} ((\text{the function sec}) \cdot f_1) - \text{id}_Z)$ ,
- (iv) for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a \cdot x$  and  $a \neq 0$ ,
- (v)  $Z = \text{dom } f$ , and
- (vi)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = (\frac{1}{a} ((\text{the function sec}) \cdot f_1) - \text{id}_Z)(\text{sup } A) - (\frac{1}{a} ((\text{the function sec}) \cdot f_1) - \text{id}_Z)(\text{inf } A).$$

(31) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function cos})(a \cdot x) - (\text{the function sin})(a \cdot x)^2}{(\text{the function sin})(a \cdot x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((-\frac{1}{a}) ((\text{the function cosec}) \cdot f_1) - \text{id}_Z)$ ,
- (iv) for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a \cdot x$  and  $a \neq 0$ ,
- (v)  $Z = \text{dom } f$ , and
- (vi)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((-\frac{1}{a}) ((\text{the function cosec}) \cdot f_1) - \text{id}_Z)(\text{sup } A) - ((-\frac{1}{a}) ((\text{the function cosec}) \cdot f_1) - \text{id}_Z)(\text{inf } A).$$

(32) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{x} + \frac{(\text{the function ln})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function sec}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((\text{the function ln}) (\text{the function sec}))(\text{sup } A) - ((\text{the function ln}) (\text{the function sec}))(\text{inf } A).$$

(33) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{x} - \frac{(\text{the function ln})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function cosec}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((\text{the function ln}) (\text{the function cosec}))(\text{sup } A) - ((\text{the function ln}) (\text{the function cosec}))(\text{inf } A).$$

(34) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{(\text{the function } \cos)(x)^2} - \frac{(\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function } \sec)$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x) dx = (-\frac{1}{\text{id}_Z} \text{ the function } \sec)(\text{sup } A) - (-\frac{1}{\text{id}_Z} \text{ the function } \sec)(\text{inf } A).$$

(35) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{(\text{the function } \sin)(x)^2} + \frac{(\text{the function } \cos)(x)}{(\text{the function } \sin)(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}(\frac{1}{\text{id}_Z} \text{ the function } \text{cosec})$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x) dx = (-\frac{1}{\text{id}_Z} \text{ the function } \text{cosec})(\text{sup } A) - (-\frac{1}{\text{id}_Z} \text{ the function } \text{cosec})(\text{inf } A).$$

(36) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function } \cos)(x) \cdot (\text{the function } \sin)((\text{the function } \sin)(x))}{(\text{the function } \cos)((\text{the function } \sin)(x))^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function } \sec) \cdot (\text{the function } \sin))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x) dx = ((\text{the function } \sec) \cdot (\text{the function } \sin))(\text{sup } A) - ((\text{the function } \sec) \cdot (\text{the function } \sin))(\text{inf } A).$$

(37) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{(\text{the function } \sin)(x) \cdot (\text{the function } \sin)((\text{the function } \cos)(x))}{(\text{the function } \cos)((\text{the function } \cos)(x))^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function } \sec) \cdot (\text{the function } \cos))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x) dx = (-(\text{the function } \sec) \cdot (\text{the function } \cos))(\text{sup } A) - (-(\text{the function } \sec) \cdot (\text{the function } \cos))(\text{inf } A).$$

(38) Suppose that

- (i)  $A \subseteq Z$ ,



- (ii) for every  $x$  such that  $x \in Z$  holds
 
$$f(x) = \frac{(\text{the function } \cos)(x) \cdot (\text{the function } \cos)((\text{the function } \sin)(x))}{(\text{the function } \sin)((\text{the function } \sin)(x))^2},$$
- (iii)  $Z \subseteq \text{dom}((\text{the function } \text{cosec}) \cdot (\text{the function } \sin)),$
- (iv)  $Z = \text{dom } f,$  and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function cosec)} \cdot \text{(the function sin)})(\text{sup } A) - (-\text{(the function cosec)} \cdot \text{(the function sin)})(\text{inf } A).$

(39) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds
 
$$f(x) = \frac{(\text{the function } \sin)(x) \cdot (\text{the function } \cos)((\text{the function } \cos)(x))}{(\text{the function } \sin)((\text{the function } \cos)(x))^2},$$
- (iii)  $Z \subseteq \text{dom}((\text{the function } \text{cosec}) \cdot (\text{the function } \cos)),$
- (iv)  $Z = \text{dom } f,$  and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function cosec}) \cdot \text{(the function cos)})(\text{sup } A) - ((\text{the function cosec}) \cdot \text{(the function cos)})(\text{inf } A).$

(40) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds
 
$$f(x) = \frac{(\text{the function } \sin)((\text{the function } \tan)(x))}{(\text{the function } \cos)((\text{the function } \tan)(x))^2},$$
- (iii)  $Z \subseteq \text{dom}((\text{the function } \text{sec}) \cdot (\text{the function } \tan)),$
- (iv)  $Z = \text{dom } f,$  and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function sec}) \cdot \text{(the function tan)})(\text{sup } A) - ((\text{the function sec}) \cdot \text{(the function tan)})(\text{inf } A).$

(41) Suppose that

- (i)  $A \subseteq Z,$
- (ii) for every  $x$  such that  $x \in Z$  holds
 
$$f(x) = \frac{(\text{the function } \sin)((\text{the function } \cot)(x))}{(\text{the function } \cos)((\text{the function } \cot)(x))^2},$$
- (iii)  $Z \subseteq \text{dom}((\text{the function } \text{sec}) \cdot (\text{the function } \cot)),$
- (iv)  $Z = \text{dom } f,$  and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function sec)} \cdot \text{(the function cot)})(\text{sup } A) - (-\text{(the function sec)} \cdot \text{(the function cot)})(\text{inf } A).$

(42) Suppose that

- (i)  $A \subseteq Z$ ,  
(ii) for every  $x$  such that  $x \in Z$  holds  

$$f(x) = \frac{\frac{(\text{the function } \cos)((\text{the function } \tan)(x))}{(\text{the function } \cos)(x)^2}}{(\text{the function } \sin)((\text{the function } \tan)(x))^2},$$
  
(iii)  $Z \subseteq \text{dom}((\text{the function } \text{cosec}) \cdot (\text{the function } \tan))$ ,  
(iv)  $Z = \text{dom } f$ , and  
(v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = (-(\text{the function } \text{cosec}) \cdot (\text{the function } \tan))(\text{sup } A) -$$

$$(-(\text{the function } \text{cosec}) \cdot (\text{the function } \tan))(\text{inf } A).$$

(43) Suppose that

- (i)  $A \subseteq Z$ ,  
(ii) for every  $x$  such that  $x \in Z$  holds  

$$f(x) = \frac{\frac{(\text{the function } \cos)((\text{the function } \cot)(x))}{(\text{the function } \sin)(x)^2}}{(\text{the function } \sin)((\text{the function } \cot)(x))^2},$$
  
(iii)  $Z \subseteq \text{dom}((\text{the function } \text{cosec}) \cdot (\text{the function } \cot))$ ,  
(iv)  $Z = \text{dom } f$ , and  
(v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((\text{the function } \text{cosec}) \cdot (\text{the function } \cot))(\text{sup } A) - ((\text{the func-}$$

$$\text{tion } \text{cosec}) \cdot (\text{the function } \cot))(\text{inf } A).$$

(44) Suppose that

- (i)  $A \subseteq Z$ ,  
(ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{\frac{1}{(\text{the function } \cos)(x)^2}}{(\text{the function } \tan)(x) \cdot (\text{the function } \sin)(x)} + \frac{(\text{the function } \tan)(x) \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2},$   
(iii)  $Z \subseteq \text{dom}((\text{the function } \tan) (\text{the function } \sec))$ ,  
(iv)  $Z = \text{dom } f$ , and  
(v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = ((\text{the function } \tan) (\text{the function } \sec))(\text{sup } A) - ((\text{the function}$$

$$\tan) (\text{the function } \sec))(\text{inf } A).$$

(45) Suppose that

- (i)  $A \subseteq Z$ ,  
(ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{\frac{1}{(\text{the function } \sin)(x)^2}}{(\text{the function } \cot)(x) \cdot (\text{the function } \sin)(x)} - \frac{(\text{the function } \cot)(x) \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2},$   
(iii)  $Z \subseteq \text{dom}((\text{the function } \cot) (\text{the function } \sec))$ ,  
(iv)  $Z = \text{dom } f$ , and  
(v)  $f \upharpoonright A$  is continuous.

$$\text{Then } \int_A f(x)dx = (-(\text{the function } \cot) (\text{the function } \sec))(\text{sup } A) - (-(\text{the}$$

function cot) (the function sec))(inf A).

(46) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{\frac{(\text{the function cos})(x)^2}{(\text{the function sin})(x)}} - \frac{(\text{the function tan})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ ,

(iii)  $Z \subseteq \text{dom}((\text{the function tan}) (\text{the function cosec}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function tan}) (\text{the function cosec}))(\text{sup } A) - ((\text{the function tan}) (\text{the function cosec}))(\text{inf } A)$ .

(47) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{\frac{(\text{the function sin})(x)^2}{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}} + \frac{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$ ,

(iii)  $Z \subseteq \text{dom}((\text{the function cot}) (\text{the function cosec}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function cot) (the function cosec)})(\text{sup } A) - (-\text{(the function cot) (the function cosec)})(\text{inf } A)$ .

(48) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{1}{(\text{the function cos})((\text{the function cot})(x))^2} \cdot \frac{1}{(\text{the function sin})(x)^2},$$

(iii)  $Z \subseteq \text{dom}((\text{the function tan}) \cdot (\text{the function cot}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function tan)} \cdot (\text{the function cot}))(\text{sup } A) - (-\text{(the function tan)} \cdot (\text{the function cot}))(\text{inf } A)$ .

(49) Suppose that

(i)  $A \subseteq Z$ ,

(ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{1}{(\text{the function cos})((\text{the function tan})(x))^2} \cdot \frac{1}{(\text{the function cos})(x)^2},$$

(iii)  $Z \subseteq \text{dom}((\text{the function tan}) \cdot (\text{the function tan}))$ ,

(iv)  $Z = \text{dom } f$ , and

(v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function tan}) \cdot (\text{the function tan}))(\sup A) - ((\text{the function tan}) \cdot (\text{the function tan}))(\inf A)$ .

(50) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{1}{(\text{the function sin})((\text{the function cot})(x))^2} \cdot \frac{1}{(\text{the function sin})(x)^2},$$

- (iii)  $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function cot}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function cot}) \cdot (\text{the function cot}))(\sup A) - ((\text{the function cot}) \cdot (\text{the function cot}))(\inf A)$ .

(51) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds

$$f(x) = \frac{1}{(\text{the function sin})((\text{the function tan})(x))^2} \cdot \frac{1}{(\text{the function cos})(x)^2},$$

- (iii)  $Z \subseteq \text{dom}((\text{the function cot}) \cdot (\text{the function tan}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-(\text{the function cot}) \cdot (\text{the function tan}))(\sup A) - (-(\text{the function cot}) \cdot (\text{the function tan}))(\inf A)$ .

(52) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{(\text{the function cos})(x)^2} + \frac{1}{(\text{the function sin})(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function tan}) - (\text{the function cot}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function tan}) - (\text{the function cot}))(\sup A) - ((\text{the function tan}) - (\text{the function cot}))(\inf A)$ .

(53) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = \frac{1}{(\text{the function cos})(x)^2} - \frac{1}{(\text{the function sin})(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function tan}) + (\text{the function cot}))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function tan})+(\text{the function cot}))(\text{sup } A) - ((\text{the function tan})+(\text{the function cot}))(\text{inf } A)$ .

(54) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function cos})((\text{the function sin})(x)) \cdot (\text{the function cos})(x)$ ,
- (iii)  $Z = \text{dom } f$ , and
- (iv)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function sin}) \cdot (\text{the function sin}))(\text{sup } A) - ((\text{the function sin}) \cdot (\text{the function sin}))(\text{inf } A)$ .

(55) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function cos})((\text{the function cos})(x)) \cdot (\text{the function sin})(x)$ ,
- (iii)  $Z = \text{dom } f$ , and
- (iv)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function sin)} \cdot \text{(the function cos)})(\text{sup } A) - (-\text{(the function sin)} \cdot \text{(the function cos)})(\text{inf } A)$ .

(56) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function sin})((\text{the function sin})(x)) \cdot (\text{the function cos})(x)$ ,
- (iii)  $Z = \text{dom } f$ , and
- (iv)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function cos)} \cdot \text{(the function sin)})(\text{sup } A) - (-\text{(the function cos)} \cdot \text{(the function sin)})(\text{inf } A)$ .

(57) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function sin})((\text{the function cos})(x)) \cdot (\text{the function sin})(x)$ ,
- (iii)  $Z = \text{dom } f$ , and
- (iv)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function cos}) \cdot (\text{the function cos}))(\text{sup } A) - ((\text{the function cos}) \cdot (\text{the function cos}))(\text{inf } A)$ .

(58) Suppose that

- (i)  $A \subseteq Z$ ,

- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function } \cos)(x) + \frac{(\text{the function } \cos)(x)}{(\text{the function } \sin)(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function } \cos) (\text{the function } \cot))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = (-\text{(the function } \cos) (\text{the function } \cot))(\sup A) - (-\text{(the function } \cos) (\text{the function } \cot))(\inf A)$ .

(59) Suppose that

- (i)  $A \subseteq Z$ ,
- (ii) for every  $x$  such that  $x \in Z$  holds  $f(x) = (\text{the function } \sin)(x) + \frac{(\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$ ,
- (iii)  $Z \subseteq \text{dom}((\text{the function } \sin) (\text{the function } \tan))$ ,
- (iv)  $Z = \text{dom } f$ , and
- (v)  $f \upharpoonright A$  is continuous.

Then  $\int_A f(x)dx = ((\text{the function } \sin) (\text{the function } \tan))(\sup A) - ((\text{the function } \sin) (\text{the function } \tan))(\inf A)$ .

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*Received February 4, 2010*

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