

# On $L^p$ Space Formed by Real-Valued Partial Functions

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**Summary.** This article is the continuation of [31]. We define the set of  $L^p$  integrable functions – the set of all partial functions whose absolute value raised to the  $p$ -th power is integrable. We show that  $L^p$  integrable functions form the  $L^p$  space. We also prove Minkowski's inequality, Hölder's inequality and that  $L^p$  space is Banach space ([15], [27]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [8], [9], [10], [4], [1], [31], [6], [19], [20], [13], [28], [14], [2], [24], [3], [11], [25], [22], [21], [16], [32], [29], [23], [18], [17], [26], [30], [5], and [12].

## 1. PRELIMINARIES ON POWERS OF NUMBERS AND OPERATIONS ON REAL SEQUENCES

For simplicity, we follow the rules:  $X$  denotes a non empty set,  $x$  denotes an element of  $X$ ,  $S$  denotes a  $\sigma$ -field of subsets of  $X$ ,  $M$  denotes a  $\sigma$ -measure on  $S$ ,  $f, g, f_1, g_1$  denote partial functions from  $X$  to  $\mathbb{R}$ , and  $a, b, c$  denote real numbers.

The following propositions are true:

- (1) For all positive real numbers  $m, n$  such that  $\frac{1}{m} + \frac{1}{n} = 1$  holds  $m > 1$ .

- (2) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ ,  $A$  be an element of  $S$ , and  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $A = \text{dom } f$  and  $f$  is measurable on  $A$  and  $f$  is non-negative. Then  $\int f \, dM \in \mathbb{R}$  if and only if  $f$  is integrable on  $M$ .

Let  $r$  be a real number. We say that  $r$  is great or equal to 1 if and only if:

- (Def. 1)  $1 \leq r$ .

Let us note that every real number which is great or equal to 1 is also positive.

One can verify that there exists a real number which is great or equal to 1.

In the sequel  $k$  denotes a positive real number.

We now state several propositions:

- (3) For all real numbers  $a, b, p$  such that  $0 < p$  and  $0 \leq a < b$  holds  $a^p < b^p$ .
- (4) If  $a \geq 0$  and  $b > 0$ , then  $a^b \geq 0$ .
- (5) If  $a \geq 0$  and  $b \geq 0$  and  $c > 0$ , then  $(a \cdot b)^c = a^c \cdot b^c$ .
- (6) For all real numbers  $a, b$  and for every  $f$  such that  $f$  is non-negative and  $a > 0$  and  $b > 0$  holds  $(f^a)^b = f^{a \cdot b}$ .
- (7) For all real numbers  $a, b$  and for every  $f$  such that  $f$  is non-negative and  $a > 0$  and  $b > 0$  holds  $f^a f^b = f^{a+b}$ .
- (8)  $f^1 = f$ .
- (9) Let  $s_1, s_2$  be sequences of real numbers and  $k$  be a positive real number. Suppose that for every element  $n$  of  $\mathbb{N}$  holds  $s_1(n) = s_2(n)^k$  and  $s_2(n) \geq 0$ . Then  $s_1$  is convergent if and only if  $s_2$  is convergent.
- (10) Let  $s_3$  be a sequence of real numbers and  $n, m$  be elements of  $\mathbb{N}$ . If  $m \leq n$ , then  $|(\sum_{\alpha=0}^{\kappa} (s_3)(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} (s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa} |s_3(\alpha)|)_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} |s_3(\alpha)|)_{\kappa \in \mathbb{N}}(m)$  and  $|(\sum_{\alpha=0}^{\kappa} (s_3)(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} (s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa} |s_3(\alpha)|)_{\kappa \in \mathbb{N}}(n)$ .
- (11) Let  $s_3, s_2$  be sequences of real numbers and  $k$  be a positive real number. Suppose  $s_3$  is convergent and for every element  $n$  of  $\mathbb{N}$  holds  $s_2(n) = |\lim s_3 - s_3(n)|^k$ . Then  $s_2$  is convergent and  $\lim s_2 = 0$ .

## 2. REAL LINEAR SPACE OF $L^p$ INTEGRABLE FUNCTIONS

Next we state two propositions:

- (12) For every positive real number  $k$  and for every non empty set  $X$  holds  $(X \mapsto 0)^k = X \mapsto 0$ .
- (13) For every partial function  $f$  from  $X$  to  $\mathbb{R}$  and for every set  $D$  holds  $|f \upharpoonright D| = |f| \upharpoonright D$ .

Let us consider  $X$  and let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . Observe that  $|f|$  is non-negative.

One can prove the following two propositions:

- (14) For every partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f$  is non-negative holds  $|f| = f$ .
- (15) If  $X = \text{dom } f$  and for every  $x$  such that  $x \in \text{dom } f$  holds  $0 = f(x)$ , then  $f$  is integrable on  $M$  and  $\int f \, dM = 0$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $L^p \text{ functions}(M, k)$  yielding a non empty subset of  $\text{PFunct}_{\text{RLS}} X$  is defined by the condition (Def. 2).

(Def. 2)  $L^p \text{ functions}(M, k) = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{R}: \bigvee_{E_1: \text{element of } S} (M(E_1^c) = 0 \wedge \text{dom } f = E_1 \wedge f \text{ is measurable on } E_1 \wedge |f|^k \text{ is integrable on } M)\}$ .

Next we state a number of propositions:

- (16) For all real numbers  $a, b, k$  such that  $k > 0$  holds  $|a + b|^k \leq (|a| + |b|)^k$  and  $(|a| + |b|)^k \leq (2 \cdot \max(|a|, |b|))^k$  and  $|a + b|^k \leq (2 \cdot \max(|a|, |b|))^k$ .
- (17) For all real numbers  $a, b, k$  such that  $a \geq 0$  and  $b \geq 0$  and  $k > 0$  holds  $(\max(a, b))^k \leq a^k + b^k$ .
- (18) For every partial function  $f$  from  $X$  to  $\mathbb{R}$  and for all real numbers  $a, b$  such that  $b > 0$  holds  $|a|^b |f|^b = |a f|^b$ .
- (19) Let  $f$  be a partial function from  $X$  to  $\mathbb{R}$  and  $a, b$  be real numbers. If  $a > 0$  and  $b > 0$ , then  $a^b |f|^b = (a |f|)^b$ .
- (20) For every partial function  $f$  from  $X$  to  $\mathbb{R}$  and for every real number  $k$  and for every set  $E$  holds  $(f \upharpoonright E)^k = f^k \upharpoonright E$ .
- (21) For all real numbers  $a, b, k$  such that  $k > 0$  holds  $|a+b|^k \leq 2^k \cdot (|a|^k + |b|^k)$ .
- (22) Let  $k$  be a positive real number and  $f, g$  be partial functions from  $X$  to  $\mathbb{R}$ . Suppose  $f, g \in L^p \text{ functions}(M, k)$ . Then  $|f|^k$  is integrable on  $M$  and  $|g|^k$  is integrable on  $M$  and  $|f|^k + |g|^k$  is integrable on  $M$ .
- (23)  $X \mapsto 0$  is a partial function from  $X$  to  $\mathbb{R}$  and  $X \mapsto 0 \in L^p \text{ functions}(M, k)$ .
- (24) Let  $k$  be a real number. Suppose  $k > 0$ . Let  $f, g$  be partial functions from  $X$  to  $\mathbb{R}$  and  $x$  be an element of  $X$ . If  $x \in \text{dom } f \cap \text{dom } g$ , then  $|f + g|^k(x) \leq (2^k (|f|^k + |g|^k))(x)$ .
- (25) If  $f, g \in L^p \text{ functions}(M, k)$ , then  $f + g \in L^p \text{ functions}(M, k)$ .
- (26) If  $f \in L^p \text{ functions}(M, k)$ , then  $a f \in L^p \text{ functions}(M, k)$ .
- (27) If  $f, g \in L^p \text{ functions}(M, k)$ , then  $f - g \in L^p \text{ functions}(M, k)$ .
- (28) If  $f \in L^p \text{ functions}(M, k)$ , then  $|f| \in L^p \text{ functions}(M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. Note that  $L^p \text{ functions}(M, k)$  is multiplicatively-closed and add closed.

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. One can check that  $\langle L^p \text{ functions}(M, k), 0_{\text{PFunct}_{\text{RLS}} X} (\in L^p \text{ functions}(M, k)), \text{add} | (L^p \text{ functions}(M, k), \text{PFunct}_{\text{RLS}} X), \cdot_{L^p \text{ functions}(M, k)} \rangle$  is Abelian, add-associative, and real linear space-like.

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{RLSp LpFunct}(M, k)$  yields a strict Abelian add-associative real linear space-like non empty RLS structure and is defined by:

(Def. 3)  $\text{RLSp LpFunct}(M, k) = \langle L^p \text{ functions}(M, k), 0_{\text{PFunct}_{\text{RLS}} X} (\in L^p \text{ functions}(M, k)), \text{add} | (L^p \text{ functions}(M, k), \text{PFunct}_{\text{RLS}} X), \cdot_{L^p \text{ functions}(M, k)} \rangle$ .

### 3. PRELIMINARIES ON REAL NORMED SPACE OF $L^p$ INTEGRABLE FUNCTIONS

In the sequel  $v, u$  are vectors of  $\text{RLSp LpFunct}(M, k)$ .

We now state three propositions:

$$(29) \quad (v) + (u) = v + u.$$

$$(30) \quad a(u) = a \cdot u.$$

(31) Suppose  $f = u$ . Then

$$(i) \quad u + (-1) \cdot u = (X \mapsto 0) \upharpoonright \text{dom } f, \text{ and}$$

$$(ii) \quad \text{there exist partial functions } v, g \text{ from } X \text{ to } \mathbb{R} \text{ such that } v, g \in L^p \text{ functions}(M, k) \text{ and } v = u + (-1) \cdot u \text{ and } g = X \mapsto 0 \text{ and } v \stackrel{M}{\text{a.e.}} g.$$

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{AlmostZeroLpFunctions}(M, k)$  yielding a non empty subset of  $\text{RLSp LpFunct}(M, k)$  is defined by:

(Def. 4)  $\text{AlmostZeroLpFunctions}(M, k) = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{R}: f \in L^p \text{ functions}(M, k) \wedge f \stackrel{M}{\text{a.e.}} X \mapsto 0\}$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. One can check that  $\text{AlmostZeroLpFunctions}(M, k)$  is add closed and multiplicatively-closed.

Next we state the proposition

$$(32) \quad 0_{\text{RLSp LpFunct}(M, k)} = X \mapsto 0 \text{ and } 0_{\text{RLSp LpFunct}(M, k)} \in \text{AlmostZeroLpFunctions}(M, k).$$

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{RLSpAlmostZeroLpFunctions}(M, k)$  yielding a non empty RLS structure is defined by:

(Def. 5)  $\text{RLSpAlmostZeroLpFunctions}(M, k) = \langle \text{AlmostZeroLpFunctions}(M, k), 0_{\text{RLSp LpFunct}(M, k)} (\in \text{AlmostZeroLpFunctions}(M, k)), \text{add} | (\text{AlmostZeroLp}$

Functions( $M, k$ ),  $\text{RLSp LpFunc}(\mathcal{M}, k)$ ,  $\cdot$   $\text{AlmostZeroLpFunctions}(\mathcal{M}, k)$ ).

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. Observe that  $\text{RLSp LpFunc}(\mathcal{M}, k)$  is strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel  $v, u$  are vectors of  $\text{RLSpAlmostZeroLpFunctions}(\mathcal{M}, k)$ .

One can prove the following two propositions:

$$(33) \quad (v) + (u) = v + u.$$

$$(34) \quad a(u) = a \cdot u.$$

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ , and let  $k$  be a positive real number. The functor a.e.-eq-class  $L^p(f, M, k)$  yields a subset of  $L^p$  functions( $M, k$ ) and is defined as follows:

(Def. 6) a.e.-eq-class  $L^p(f, M, k) = \{h; h \text{ ranges over partial functions from } X \text{ to } \mathbb{R}; h \in L^p \text{ functions}(M, k) \wedge f =_{\text{a.e.}}^M h\}$ .

Next we state a number of propositions:

(35) If  $f \in L^p$  functions( $M, k$ ), then there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $\text{dom } f = E$  and  $f$  is measurable on  $E$ .

(36) If  $g \in L^p$  functions( $M, k$ ) and  $g =_{\text{a.e.}}^M f$ , then  $g \in$  a.e.-eq-class  $L^p(f, M, k)$ .

(37) Suppose there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $g \in$  a.e.-eq-class  $L^p(f, M, k)$ . Then  $g =_{\text{a.e.}}^M f$  and  $f \in L^p$  functions( $M, k$ ).

(38) If  $f \in L^p$  functions( $M, k$ ), then  $f \in$  a.e.-eq-class  $L^p(f, M, k)$ .

(39) Suppose there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } g$  and  $g$  is measurable on  $E$  and a.e.-eq-class  $L^p(f, M, k) \neq \emptyset$  and a.e.-eq-class  $L^p(f, M, k) =$  a.e.-eq-class  $L^p(g, M, k)$ . Then  $f =_{\text{a.e.}}^M g$ .

(40) Suppose  $f \in L^p$  functions( $M, k$ ) and there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } g$  and  $g$  is measurable on  $E$  and a.e.-eq-class  $L^p(f, M, k) =$  a.e.-eq-class  $L^p(g, M, k)$ . Then  $f =_{\text{a.e.}}^M g$ .

(41) If  $f =_{\text{a.e.}}^M g$ , then a.e.-eq-class  $L^p(f, M, k) =$  a.e.-eq-class  $L^p(g, M, k)$ .

(42) If  $f =_{\text{a.e.}}^M g$ , then a.e.-eq-class  $L^p(f, M, k) =$  a.e.-eq-class  $L^p(g, M, k)$ .

(43) If  $f \in L^p$  functions( $M, k$ ) and  $g \in$  a.e.-eq-class  $L^p(f, M, k)$ , then a.e.-eq-class  $L^p(f, M, k) =$  a.e.-eq-class  $L^p(g, M, k)$ .

(44) Suppose that there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } f_1$  and  $f_1$  is measurable on  $E$  and there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } g$  and  $g$  is measurable on  $E$  and there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $E = \text{dom } g_1$  and  $g_1$  is measurable on

$E$  and a.e-eq-class  $L^p(f, M, k)$  is non empty and a.e-eq-class  $L^p(g, M, k)$  is non empty and a.e-eq-class  $L^p(f, M, k) = \text{a.e-eq-class } L^p(f_1, M, k)$  and a.e-eq-class  $L^p(g, M, k) = \text{a.e-eq-class } L^p(g_1, M, k)$ . Then a.e-eq-class  $L^p(f + g, M, k) = \text{a.e-eq-class } L^p(f_1 + g_1, M, k)$ .

- (45) If  $f, f_1, g, g_1 \in L^p$  functions( $M, k$ ) and a.e-eq-class  $L^p(f, M, k) = \text{a.e-eq-class } L^p(f_1, M, k)$  and a.e-eq-class  $L^p(g, M, k) = \text{a.e-eq-class } L^p(g_1, M, k)$ , then a.e-eq-class  $L^p(f + g, M, k) = \text{a.e-eq-class } L^p(f_1 + g_1, M, k)$ .

(46) Suppose that

- (i) there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $\text{dom } f = E$  and  $f$  is measurable on  $E$ ,
- (ii) there exists an element  $E$  of  $S$  such that  $M(E^c) = 0$  and  $\text{dom } g = E$  and  $g$  is measurable on  $E$ ,
- (iii) a.e-eq-class  $L^p(f, M, k)$  is non empty, and
- (iv) a.e-eq-class  $L^p(f, M, k) = \text{a.e-eq-class } L^p(g, M, k)$ .

Then a.e-eq-class  $L^p(a f, M, k) = \text{a.e-eq-class } L^p(a g, M, k)$ .

- (47) If  $f, g \in L^p$  functions( $M, k$ ) and a.e-eq-class  $L^p(f, M, k) = \text{a.e-eq-class } L^p(g, M, k)$ , then a.e-eq-class  $L^p(a f, M, k) = \text{a.e-eq-class } L^p(a g, M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{CosetSet}(M, k)$  yielding a non empty family of subsets of  $L^p$  functions( $M, k$ ) is defined by:

- (Def. 7)  $\text{CosetSet}(M, k) = \{\text{a.e-eq-class } L^p(f, M, k); f \text{ ranges over partial functions from } X \text{ to } \mathbb{R}: f \in L^p \text{ functions}(M, k)\}$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{addCoset}(M, k)$  yields a binary operation on  $\text{CosetSet}(M, k)$  and is defined by the condition (Def. 8).

- (Def. 8) Let  $A, B$  be elements of  $\text{CosetSet}(M, k)$  and  $a, b$  be partial functions from  $X$  to  $\mathbb{R}$ . If  $a \in A$  and  $b \in B$ , then  $(\text{addCoset}(M, k))(A, B) = \text{a.e-eq-class } L^p(a + b, M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{zeroCoset}(M, k)$  yields an element of  $\text{CosetSet}(M, k)$  and is defined as follows:

- (Def. 9)  $\text{zeroCoset}(M, k) = \text{a.e-eq-class } L^p(X \mapsto 0, M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{lmultCoset}(M, k)$  yielding a function from  $\mathbb{R} \times \text{CosetSet}(M, k)$  into  $\text{CosetSet}(M, k)$  is defined by the condition (Def. 10).

(Def. 10) Let  $z$  be an element of  $\mathbb{R}$ ,  $A$  be an element of  $\text{CosetSet}(M, k)$ , and  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . If  $f \in A$ , then  $(\text{lmultCoset}(M, k))(z, A) = \text{a.e-eq-class } L^p(zf, M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $\text{Pre-}L^p\text{-Space}(M, k)$  yielding a strict RLS structure is defined by the conditions (Def. 11).

- (Def. 11)(i) The carrier of  $\text{Pre-}L^p\text{-Space}(M, k) = \text{CosetSet}(M, k)$ ,  
(ii) the addition of  $\text{Pre-}L^p\text{-Space}(M, k) = \text{addCoset}(M, k)$ ,  
(iii)  $0_{\text{Pre-}L^p\text{-Space}(M, k)} = \text{zeroCoset}(M, k)$ , and  
(iv) the external multiplication of  $\text{Pre-}L^p\text{-Space}(M, k) = \text{lmultCoset}(M, k)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. Observe that  $\text{Pre-}L^p\text{-Space}(M, k)$  is non empty.

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. Observe that  $\text{Pre-}L^p\text{-Space}(M, k)$  is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

#### 4. REAL NORMED SPACE OF $L^p$ INTEGRABLE FUNCTIONS

The following propositions are true:

- (48) If  $f, g \in L^p \text{ functions}(M, k)$  and  $f \stackrel{M}{\text{a.e.}} g$ , then  $\int |f|^k dM = \int |g|^k dM$ .  
(49) If  $f \in L^p \text{ functions}(M, k)$ , then  $\int |f|^k dM \in \mathbb{R}$  and  $0 \leq \int |f|^k dM$ .  
(50) If there exists a vector  $x$  of  $\text{Pre-}L^p\text{-Space}(M, k)$  such that  $f, g \in x$ , then  $f \stackrel{M}{\text{a.e.}} g$  and  $f, g \in L^p \text{ functions}(M, k)$ .  
(51) Let  $k$  be a positive real number. Then there exists a function  $N_1$  from the carrier of  $\text{Pre-}L^p\text{-Space}(M, k)$  into  $\mathbb{R}$  such that for every point  $x$  of  $\text{Pre-}L^p\text{-Space}(M, k)$  holds there exists a partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f \in x$  and there exists a real number  $r$  such that  $r = \int |f|^k dM$  and  $N_1(x) = r^{\frac{1}{k}}$ .

In the sequel  $x$  denotes a point of  $\text{Pre-}L^p\text{-Space}(M, k)$ .

We now state two propositions:

- (52) If  $f \in x$ , then  $|f|^k$  is integrable on  $M$  and  $f \in L^p \text{ functions}(M, k)$ .  
(53) If  $f, g \in x$ , then  $f \stackrel{M}{\text{a.e.}} g$  and  $\int |f|^k dM = \int |g|^k dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $L^p\text{-Norm}(M, k)$  yielding a function from the carrier of  $\text{Pre-}L^p\text{-Space}(M, k)$  into  $\mathbb{R}$  is defined by the condition (Def. 12).

(Def. 12) Let  $x$  be a point of  $\text{Pre-}L^p\text{-Space}(M, k)$ . Then there exists a partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f \in x$  and there exists a real number  $r$  such that  $r = \int |f|^k dM$  and  $(L^p\text{-Norm}(M, k))(x) = r^{\frac{1}{k}}$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $k$  be a positive real number. The functor  $L^p\text{-Space}(M, k)$  yields a non empty normed structure and is defined by:

(Def. 13)  $L^p\text{-Space}(M, k) = \langle \text{the carrier of Pre-}L^p\text{-Space}(M, k), \text{ the zero of Pre-}L^p\text{-Space}(M, k), \text{ the addition of Pre-}L^p\text{-Space}(M, k), \text{ the external multiplication of Pre-}L^p\text{-Space}(M, k), L^p\text{-Norm}(M, k) \rangle$ .

In the sequel  $x, y$  denote points of  $L^p\text{-Space}(M, k)$ .

One can prove the following propositions:

(54)(i) There exists a partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f \in L^p\text{ functions}(M, k)$  and  $x = \text{a.e-eq-class } L^p(f, M, k)$ , and

(ii) for every partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f \in x$  there exists a real number  $r$  such that  $0 \leq r = \int |f|^k dM$  and  $\|x\| = r^{\frac{1}{k}}$ .

(55) If  $f \in x$  and  $g \in y$ , then  $f + g \in x + y$  and if  $f \in x$ , then  $a f \in a \cdot x$ .

(56) If  $f \in x$ , then  $x = \text{a.e-eq-class } L^p(f, M, k)$  and there exists a real number  $r$  such that  $0 \leq r = \int |f|^k dM$  and  $\|x\| = r^{\frac{1}{k}}$ .

(57)  $X \mapsto 0 \in \text{the } L^1 \text{ functions of } M$ .

(58) If  $f \in L^p\text{ functions}(M, k)$  and  $\int |f|^k dM = 0$ , then  $f =_{\text{a.e.}}^M X \mapsto 0$ .

(59)  $\int |X \mapsto 0|^k dM = 0$ .

(60) Let  $m, n$  be positive real numbers. Suppose  $\frac{1}{m} + \frac{1}{n} = 1$  and  $f \in L^p\text{ functions}(M, m)$  and  $g \in L^p\text{ functions}(M, n)$ . Then  $f g \in \text{the } L^1 \text{ functions of } M$  and  $f g$  is integrable on  $M$ .

(61) Let  $m, n$  be positive real numbers. Suppose  $\frac{1}{m} + \frac{1}{n} = 1$  and  $f \in L^p\text{ functions}(M, m)$  and  $g \in L^p\text{ functions}(M, n)$ . Then there exists a real number  $r_1$  such that  $r_1 = \int |f|^m dM$  and there exists a real number  $r_2$  such that  $r_2 = \int |g|^n dM$  and  $\int |f g| dM \leq r_1^{\frac{1}{m}} \cdot r_2^{\frac{1}{n}}$ .

(62) Let  $m$  be a positive real number and  $r_1, r_2, r_3$  be elements of  $\mathbb{R}$ . Suppose  $1 \leq m$  and  $f, g \in L^p\text{ functions}(M, m)$  and  $r_1 = \int |f|^m dM$  and  $r_2 = \int |g|^m dM$  and  $r_3 = \int |f + g|^m dM$ . Then  $r_3^{\frac{1}{m}} \leq r_1^{\frac{1}{m}} + r_2^{\frac{1}{m}}$ .

Let  $k$  be a great or equal to 1 real number, let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . Note that  $L^p\text{-Space}(M, k)$  is reflexive, discernible, real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.



5. PRELIMINARIES ON COMPLETENESS OF  $L^p$  SPACE

The following propositions are true:

- (63) Let  $S_1$  be a sequence of  $L^p$ -Space( $M, k$ ). Then there exists a sequence  $F_1$  of partial functions from  $X$  into  $\mathbb{R}$  such that for every element  $n$  of  $\mathbb{N}$  holds  
 $F_1(n) \in L^p \text{ functions}(M, k)$  and  $F_1(n) \in S_1(n)$  and  $S_1(n) =$  a.e-eq-class  $L^p(F_1(n), M, k)$  and there exists a real number  $r$  such that  $r = \int |F_1(n)|^k dM$  and  $\|S_1(n)\| = r^{\frac{1}{k}}$ .
- (64) Let  $S_1$  be a sequence of  $L^p$ -Space( $M, k$ ). Then there exists a sequence  $F_1$  of partial functions from  $X$  into  $\mathbb{R}$  with the same dom such that for every element  $n$  of  $\mathbb{N}$  holds  
 $F_1(n) \in L^p \text{ functions}(M, k)$  and  $F_1(n) \in S_1(n)$  and  $S_1(n) =$  a.e-eq-class  $L^p(F_1(n), M, k)$  and there exists a real number  $r$  such that  $0 \leq r = \int |F_1(n)|^k dM$  and  $\|S_1(n)\| = r^{\frac{1}{k}}$ .
- (65) Let  $X$  be a real normed space,  $S_1$  be a sequence of  $X$ , and  $S_0$  be a point of  $X$ . If  $\|S_1 - S_0\|$  is convergent and  $\lim \|S_1 - S_0\| = 0$ , then  $S_1$  is convergent and  $\lim S_1 = S_0$ .
- (66) Let  $X$  be a real normed space and  $S_1$  be a sequence of  $X$ . Suppose  $S_1$  is Cauchy sequence by norm. Then there exists an increasing function  $N$  from  $\mathbb{N}$  into  $\mathbb{N}$  such that for all elements  $i, j$  of  $\mathbb{N}$  if  $j \geq N(i)$ , then  $\|S_1(j) - S_1(N(i))\| < 2^{-i}$ .
- (67) Let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ . Suppose that for every natural number  $m$  holds  $F(m) \in L^p \text{ functions}(M, k)$ . Let  $m$  be a natural number. Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \in L^p \text{ functions}(M, k)$ .
- (68) Let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ . Suppose that for every natural number  $m$  holds  $F(m)$  is non-negative. Let  $m$  be a natural number. Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is non-negative.
- (69) Let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ ,  $x$  be an element of  $X$ , and  $n, m$  be natural numbers. Suppose  $F$  has the same dom and  $x \in \text{dom } F(0)$  and for every natural number  $k$  holds  $F(k)$  is non-negative and  $n \leq m$ . Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$ .
- (70) For every sequence  $F$  of partial functions from  $X$  into  $\mathbb{R}$  such that  $F$  has the same dom holds  $|F|$  has the same dom.
- (71) Let  $k$  be a great or equal to 1 real number and  $S_1$  be a sequence of  $L^p$ -Space( $M, k$ ). If  $S_1$  is Cauchy sequence by norm, then  $S_1$  is convergent.

Let us consider  $X, S, M$  and let  $k$  be a great or equal to 1 real number. Observe that  $L^p$ -Space( $M, k$ ) is complete.

6. RELATIONS BETWEEN  $L^1$  SPACE AND  $L^p$  SPACE

One can prove the following propositions:

- (72) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $\text{CosetSet } M = \text{CosetSet}(M, 1)$ .
- (73) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $\text{addCoset } M = \text{addCoset}(M, 1)$ .
- (74) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $\text{zeroCoset } M = \text{zeroCoset}(M, 1)$ .
- (75) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $\text{lmultCoset } M = \text{lmultCoset}(M, 1)$ .
- (76) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $\text{pre-}L\text{-Space } M = \text{Pre-}L^p\text{-Space}(M, 1)$ .
- (77) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $L^1\text{-Norm}(M) = L^p\text{-Norm}(M, 1)$ .
- (78) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $M$  be a  $\sigma$ -measure on  $S$ . Then  $L^1\text{-Space}(M) = L^p\text{-Space}(M, 1)$ .

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