

Miscellaneous Facts about Open Functions and Continuous Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. In this article we give definitions of open functions and continuous functions formulated in terms of “balls” of given topological spaces.

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The notation and terminology used here have been introduced in the following papers: [6], [4], [5], [8], [1], [2], [3], [10], [11], [12], [7], [9], and [13].

1. OPEN FUNCTIONS

We adopt the following rules: n, m are elements of \mathbb{N} , T is a non empty topological space, and M, M_1, M_2 are non empty metric spaces.

The following propositions are true:

- (1) Let A, B, S, T be topological spaces, f be a function from A into S , and g be a function from B into T . Suppose that
 - (i) the topological structure of $A =$ the topological structure of B ,
 - (ii) the topological structure of $S =$ the topological structure of T ,
 - (iii) $f = g$, and
 - (iv) f is open.

Then g is open.

- (2) Let P be a subset of \mathcal{E}_T^m . Then P is open if and only if for every point p of \mathcal{E}_T^m such that $p \in P$ there exists a positive real number r such that $\text{Ball}(p, r) \subseteq P$.

- (3) Let X, Y be non empty topological spaces and f be a function from X into Y . Then f is open if and only if for every point p of X and for every open subset V of X such that $p \in V$ there exists an open subset W of Y such that $f(p) \in W$ and $W \subseteq f^\circ V$.
- (4) Let f be a function from T into M_{top} . Then f is open if and only if for every point p of T and for every open subset V of T and for every point q of M such that $q = f(p)$ and $p \in V$ there exists a positive real number r such that $\text{Ball}(q, r) \subseteq f^\circ V$.
- (5) Let f be a function from M_{top} into T . Then f is open if and only if for every point p of M and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^\circ \text{Ball}(p, r)$.
- (6) Let f be a function from $(M_1)_{\text{top}}$ into $(M_2)_{\text{top}}$. Then f is open if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $\text{Ball}(q, s) \subseteq f^\circ \text{Ball}(p, r)$.
- (7) Let f be a function from T into \mathcal{E}_T^m . Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $\text{Ball}(f(p), r) \subseteq f^\circ V$.
- (8) Let f be a function from \mathcal{E}_T^m into T . Then f is open if and only if for every point p of \mathcal{E}_T^m and for every positive real number r there exists an open subset W of T such that $f(p) \in W$ and $W \subseteq f^\circ \text{Ball}(p, r)$.
- (9) Let f be a function from \mathcal{E}_T^m into \mathcal{E}_T^n . Then f is open if and only if for every point p of \mathcal{E}_T^m and for every positive real number r there exists a positive real number s such that $\text{Ball}(f(p), s) \subseteq f^\circ \text{Ball}(p, r)$.
- (10) Let f be a function from T into \mathbb{R}^1 . Then f is open if and only if for every point p of T and for every open subset V of T such that $p \in V$ there exists a positive real number r such that $]f(p) - r, f(p) + r[\subseteq f^\circ V$.
- (11) Let f be a function from \mathbb{R}^1 into T . Then f is open if and only if for every point p of \mathbb{R}^1 and for every positive real number r there exists an open subset V of T such that $f(p) \in V$ and $V \subseteq f^\circ]p - r, p + r[$.
- (12) Let f be a function from \mathbb{R}^1 into \mathbb{R}^1 . Then f is open if and only if for every point p of \mathbb{R}^1 and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^\circ]p - r, p + r[$.
- (13) Let f be a function from \mathcal{E}_T^m into \mathbb{R}^1 . Then f is open if and only if for every point p of \mathcal{E}_T^m and for every positive real number r there exists a positive real number s such that $]f(p) - s, f(p) + s[\subseteq f^\circ \text{Ball}(p, r)$.
- (14) Let f be a function from \mathbb{R}^1 into \mathcal{E}_T^m . Then f is open if and only if for every point p of \mathbb{R}^1 and for every positive real number r there exists a positive real number s such that $\text{Ball}(f(p), s) \subseteq f^\circ]p - r, p + r[$.

2. CONTINUOUS FUNCTIONS

Next we state a number of propositions:

- (15) Let f be a function from T into M_{top} . Then f is continuous if and only if for every point p of T and for every point q of M and for every positive real number r such that $q = f(p)$ there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq \text{Ball}(q, r)$.
- (16) Let f be a function from M_{top} into T . Then f is continuous if and only if for every point p of M and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ \text{Ball}(p, s) \subseteq V$.
- (17) Let f be a function from $(M_1)_{\text{top}}$ into $(M_2)_{\text{top}}$. Then f is continuous if and only if for every point p of M_1 and for every point q of M_2 and for every positive real number r such that $q = f(p)$ there exists a positive real number s such that $f^\circ \text{Ball}(p, s) \subseteq \text{Ball}(q, r)$.
- (18) Let f be a function from T into \mathcal{E}_T^m . Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq \text{Ball}(f(p), r)$.
- (19) Let f be a function from \mathcal{E}_T^m into T . Then f is continuous if and only if for every point p of \mathcal{E}_T^m and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ \text{Ball}(p, s) \subseteq V$.
- (20) Let f be a function from \mathcal{E}_T^m into \mathcal{E}_T^n . Then f is continuous if and only if for every point p of \mathcal{E}_T^m and for every positive real number r there exists a positive real number s such that $f^\circ \text{Ball}(p, s) \subseteq \text{Ball}(f(p), r)$.
- (21) Let f be a function from T into \mathbb{R}^1 . Then f is continuous if and only if for every point p of T and for every positive real number r there exists an open subset W of T such that $p \in W$ and $f^\circ W \subseteq]f(p) - r, f(p) + r[$.
- (22) Let f be a function from \mathbb{R}^1 into T . Then f is continuous if and only if for every point p of \mathbb{R}^1 and for every open subset V of T such that $f(p) \in V$ there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq V$.
- (23) Let f be a function from \mathbb{R}^1 into \mathbb{R}^1 . Then f is continuous if and only if for every point p of \mathbb{R}^1 and for every positive real number r there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq]f(p) - r, f(p) + r[$.
- (24) Let f be a function from \mathcal{E}_T^m into \mathbb{R}^1 . Then f is continuous if and only if for every point p of \mathcal{E}_T^m and for every positive real number r there exists a positive real number s such that $f^\circ \text{Ball}(p, s) \subseteq]f(p) - r, f(p) + r[$.
- (25) Let f be a function from \mathbb{R}^1 into \mathcal{E}_T^m . Then f is continuous if and only if for every point p of \mathbb{R}^1 and for every positive real number r there exists a positive real number s such that $f^\circ]p - s, p + s[\subseteq \text{Ball}(f(p), r)$.

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