

On the Continuity of Some Functions

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Summary. We prove that basic arithmetic operations preserve continuity of functions.

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The terminology and notation used here have been introduced in the following articles: [20], [1], [6], [13], [4], [7], [19], [8], [9], [5], [21], [2], [3], [10], [18], [25], [26], [23], [12], [22], [24], [14], [16], [17], [15], and [11].

1. PRELIMINARIES

For simplicity, we adopt the following rules: x, X are sets, i, n, m are natural numbers, r, s are real numbers, c, c_1, c_2, d are complex numbers, f, g are complex-valued functions, g_1 is an n -element complex-valued finite sequence, f_1 is an n -element real-valued finite sequence, T is a non empty topological space, and p is an element of \mathcal{E}_T^n .

Let R be a binary relation and let X be an empty set. Observe that $R^\circ X$ is empty and $R^{-1}(X)$ is empty.

Let A be an empty set. Observe that every element of A is empty.

We now state the proposition

- (1) For every trivial set X and for every set Y such that $X \approx Y$ holds Y is trivial.

Let r be a real number. Observe that r^2 is non negative.

Let r be a positive real number. Note that r^2 is positive.

Let us note that $\sqrt{0}$ is zero.

Let f be an empty set. Note that 2f is empty and $|f|$ is zero.

The following propositions are true:

- (2) $f(c_1 + c_2) = f c_1 + f c_2.$
- (3) $f(c_1 - c_2) = f c_1 - f c_2.$
- (4) $f/c + g/c = (f + g)/c.$
- (5) $f/c - g/c = (f - g)/c.$
- (6) If $c_1 \neq 0$ and $c_2 \neq 0$, then $f/c_1 - g/c_2 = (f c_2 - g c_1)/(c_1 \cdot c_2).$
- (7) If $c \neq 0$, then $f/c - g = (f - c g)/c.$
- (8) $(c - d) f = c f - d f.$
- (9) $(f - g)^2 = (g - f)^2.$
- (10) $(f/c)^2 = f^2/c^2.$
- (11) $|n \mapsto r - n \mapsto s| = \sqrt{n} \cdot |r - s|.$

Let us consider f, x, c . Observe that $f + \cdot (x, c)$ is complex-valued.

We now state a number of propositions:

- (12) $(\underbrace{\langle 0, \dots, 0 \rangle}_n + \cdot (x, c))^2 = \underbrace{\langle 0, \dots, 0 \rangle}_n + \cdot (x, c^2).$
- (13) If $x \in \text{Seg } n$, then $|\underbrace{\langle 0, \dots, 0 \rangle}_n + \cdot (x, r)| = |r|.$
- (14) $0_{\mathcal{E}_T^n} + \cdot (x, 0) = 0_{\mathcal{E}_T^n}.$
- (15) $f_1 \bullet (0_{\mathcal{E}_T^n} + \cdot (x, r)) = 0_{\mathcal{E}_T^n} + \cdot (x, f_1(x) \cdot r).$
- (16) $|(f_1, 0_{\mathcal{E}_T^n} + \cdot (x, r))| = f_1(x) \cdot r.$
- (17) $(g_1 + \cdot (i, c)) - g_1 = \underbrace{\langle 0, \dots, 0 \rangle}_n + \cdot (i, c - g_1(i)).$
- (18) $|\langle r \rangle| = |r|.$
- (19) Every real-valued finite sequence is a finite sequence of elements of \mathbb{R} .
- (20) For every real-valued finite sequence f such that $|f| \neq 0$ there exists a natural number i such that $i \in \text{dom } f$ and $f(i) \neq 0$.
- (21) For every real-valued finite sequence f holds $|\sum f| \leq \sum |f|.$
- (22) Let A be a non empty 1-sorted structure, B be a trivial non empty 1-sorted structure, t be a point of B , and f be a function from A into B . Then $f = A \mapsto t$.

Let n be a non zero natural number, let i be an element of $\text{Seg } n$, and let T be a real-membered non empty topological space. Note that $\text{proj}(\text{Seg } n \mapsto T, i)$ is real-valued.

Let us consider n , let p be an element of \mathcal{R}^n , and let us consider r . Then p/r is an element of \mathcal{R}^n .

One can prove the following proposition

- (23) For all points p, q of \mathcal{E}_T^m holds $p \in \text{Ball}(q, r)$ iff $-p \in \text{Ball}(-q, r).$

Let S be a 1-sorted structure. We say that S is complex-functions-membered if and only if:

(Def. 1) The carrier of S is complex-functions-membered.

We say that S is real-functions-membered if and only if:

(Def. 2) The carrier of S is real-functions-membered.

Let us consider n . One can verify that \mathcal{E}_T^n is real-functions-membered.

Let us observe that \mathcal{E}_T^0 is real-membered.

One can check that \mathcal{E}_T^0 is trivial.

Let us observe that every 1-sorted structure which is real-functions-membered is also complex-functions-membered.

Let us mention that there exists a 1-sorted structure which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered 1-sorted structure. One can check that the carrier of S is complex-functions-membered.

Let S be a real-functions-membered 1-sorted structure. Note that the carrier of S is real-functions-membered.

Let us observe that there exists a topological space which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered topological space. Observe that every subspace of S is complex-functions-membered.

Let S be a real-functions-membered topological space. One can verify that every subspace of S is real-functions-membered.

Let X be a complex-functions-membered set. The functor $(-)X$ yields a complex-functions-membered set and is defined as follows:

(Def. 3) For every complex-valued function f holds $-f \in (-)X$ iff $f \in X$.

Let us observe that the functor $(-)X$ is involutive.

Let X be an empty set. One can verify that $(-)X$ is empty.

Let X be a non empty complex-functions-membered set. Observe that $(-)X$ is non empty.

The following proposition is true

(24) Let X be a complex-functions-membered set and f be a complex-valued function. Then $-f \in X$ if and only if $f \in (-)X$.

Let X be a real-functions-membered set. One can verify that $(-)X$ is real-functions-membered.

Next we state the proposition

(25) For every subset X of \mathcal{E}_T^n holds $-X = (-)X$.

Let us consider n and let X be a subset of \mathcal{E}_T^n . Then $(-)X$ is a subset of \mathcal{E}_T^n .

Let us consider n and let X be an open subset of \mathcal{E}_T^n . Observe that $(-)X$ is open.

Let us consider n, p, x . Then $p(x)$ is an element of \mathbb{R} .

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T , and let x be a point of $R \times S$. Then $f(x)$ is a point of T .

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T , let r be a point of R , and let s be a point of S . Then $f(r, s)$ is a point of T .

Let us consider n, p, r . Then $p + r$ is a point of \mathcal{E}_T^n .

Let us consider n, p, r . Then $p - r$ is a point of \mathcal{E}_T^n .

Let us consider n, p, r . Then pr is a point of \mathcal{E}_T^n .

Let us consider n, p, r . Then p/r is a point of \mathcal{E}_T^n .

Let us consider n and let p_1, p_2 be points of \mathcal{E}_T^n . Then $p_1 p_2$ is a point of \mathcal{E}_T^n .

Let us note that the functor $p_1 p_2$ is commutative.

Let us consider n and let p be a point of \mathcal{E}_T^n . Then 2p is a point of \mathcal{E}_T^n .

Let us consider n and let p_1, p_2 be points of \mathcal{E}_T^n . Then p_1/p_2 is a point of \mathcal{E}_T^n .

Let us consider n, p, x, r . Then $p + \cdot (x, r)$ is a point of \mathcal{E}_T^n .

Next we state the proposition

- (26) For all points a, o of \mathcal{E}_T^n such that $n \neq 0$ and $a \in \text{Ball}(o, r)$ holds $|\Sigma(a - o)| < n \cdot r$.

Let us consider n . Note that \mathcal{E}^n is real-functions-membered.

One can prove the following propositions:

- (27) Let V be an add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V . Then $(v + u) - u = v$.
- (28) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V . Then $(v - u) + u = v$.
- (29) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f + c = f + (\text{dom } f \mapsto c)$.
- (30) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f - c = f - (\text{dom } f \mapsto c)$.
- (31) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f \cdot c = f \cdot (\text{dom } f \mapsto c)$.
- (32) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f/c = f/(\text{dom } f \mapsto c)$.

Let D be a complex-functions-membered set and let f, g be finite sequences of elements of D . One can verify the following observations:

- * $f + g$ is finite sequence-like,
- * $f - g$ is finite sequence-like,
- * $f \cdot g$ is finite sequence-like, and
- * f/g is finite sequence-like.

Next we state a number of propositions:

- (33) For every function f from X into \mathcal{E}_T^n holds $-f$ is a function from X into \mathcal{E}_T^n .
- (34) For every function f from \mathcal{E}_T^i into \mathcal{E}_T^n holds $f \circ -$ is a function from \mathcal{E}_T^i into \mathcal{E}_T^n .
- (35) For every function f from X into \mathcal{E}_T^n holds $f + r$ is a function from X into \mathcal{E}_T^n .
- (36) For every function f from X into \mathcal{E}_T^n holds $f - r$ is a function from X into \mathcal{E}_T^n .
- (37) For every function f from X into \mathcal{E}_T^n holds $f \cdot r$ is a function from X into \mathcal{E}_T^n .
- (38) For every function f from X into \mathcal{E}_T^n holds f/r is a function from X into \mathcal{E}_T^n .
- (39) For all functions f, g from X into \mathcal{E}_T^n holds $f + g$ is a function from X into \mathcal{E}_T^n .
- (40) For all functions f, g from X into \mathcal{E}_T^n holds $f - g$ is a function from X into \mathcal{E}_T^n .
- (41) For all functions f, g from X into \mathcal{E}_T^n holds $f \cdot g$ is a function from X into \mathcal{E}_T^n .
- (42) For all functions f, g from X into \mathcal{E}_T^n holds f/g is a function from X into \mathcal{E}_T^n .
- (43) Let f be a function from X into \mathcal{E}_T^n and g be a function from X into \mathbb{R}^1 . Then $f + g$ is a function from X into \mathcal{E}_T^n .
- (44) Let f be a function from X into \mathcal{E}_T^n and g be a function from X into \mathbb{R}^1 . Then $f - g$ is a function from X into \mathcal{E}_T^n .
- (45) Let f be a function from X into \mathcal{E}_T^n and g be a function from X into \mathbb{R}^1 . Then $f \cdot g$ is a function from X into \mathcal{E}_T^n .
- (46) Let f be a function from X into \mathcal{E}_T^n and g be a function from X into \mathbb{R}^1 . Then f/g is a function from X into \mathcal{E}_T^n .

Let n be a natural number, let T be a non empty set, let R be a real-membered set, and let f be a function from T into R . The functor $\text{incl}(f, n)$ yields a function from T into \mathcal{E}_T^n and is defined by:

(Def. 4) For every element t of T holds $(\text{incl}(f, n))(t) = n \mapsto f(t)$.

We now state several propositions:

- (47) Let R be a real-membered set, f be a function from T into R , and t be a point of T . If $x \in \text{Seg } n$, then $(\text{incl}(f, n))(t)(x) = f(t)$.
- (48) For every non empty set T and for every real-membered set R and for every function f from T into R holds $\text{incl}(f, 0) = T \mapsto 0$.
- (49) For every function f from T into \mathcal{E}_T^n and for every function g from T into \mathbb{R}^1 holds $f + g = f + \text{incl}(g, n)$.

- (50) For every function f from T into \mathcal{E}_T^n and for every function g from T into \mathbb{R}^1 holds $f - g = f - \text{incl}(g, n)$.
- (51) For every function f from T into \mathcal{E}_T^n and for every function g from T into \mathbb{R}^1 holds $f \cdot g = f \cdot \text{incl}(g, n)$.
- (52) For every function f from T into \mathcal{E}_T^n and for every function g from T into \mathbb{R}^1 holds $f/g = f/\text{incl}(g, n)$.

Let us consider n . The functor \otimes_n yields a function from $\mathcal{E}_T^n \times \mathcal{E}_T^n$ into \mathcal{E}_T^n and is defined by:

(Def. 5) For all points x, y of \mathcal{E}_T^n holds $\otimes_n(x, y) = xy$.

Next we state two propositions:

- (53) $\otimes_0 = \mathcal{E}_T^0 \times \mathcal{E}_T^0 \mapsto 0_{\mathcal{E}_T^0}$.
- (54) For all functions f, g from T into \mathcal{E}_T^n holds $f \cdot g = (\otimes_n)^\circ(f, g)$.

Let us consider m, n . The functor $\text{PROJ}(m, n)$ yields a function from \mathcal{E}_T^m into \mathbb{R}^1 and is defined as follows:

(Def. 6) For every element p of \mathcal{E}_T^m holds $(\text{PROJ}(m, n))(p) = p_n$.

One can prove the following propositions:

- (55) For every point p of \mathcal{E}_T^m such that $n \in \text{dom } p$ holds $(\text{PROJ}(m, n))^\circ \text{Ball}(p, r) =]p_n - r, p_n + r[$.
- (56) For every non zero natural number m and for every function f from T into \mathbb{R}^1 holds $f = \text{PROJ}(m, m) \cdot \text{incl}(f, m)$.

2. CONTINUITY

Let us consider T . One can check that there exists a function from T into \mathbb{R}^1 which is non-empty and continuous.

Next we state two propositions:

- (57) If $n \in \text{Seg } m$, then $\text{PROJ}(m, n)$ is continuous.
- (58) If $n \in \text{Seg } m$, then $\text{PROJ}(m, n)$ is open.

Let us consider n, T and let f be a continuous function from T into \mathbb{R}^1 . Observe that $\text{incl}(f, n)$ is continuous.

Let us consider n . One can verify that \otimes_n is continuous.

One can prove the following proposition

- (59) Let f be a function from \mathcal{E}_T^m into \mathcal{E}_T^n . Suppose f is continuous. Then $f \circ -$ is a continuous function from \mathcal{E}_T^m into \mathcal{E}_T^n .

Let us consider T and let f be a continuous function from T into \mathbb{R}^1 . Observe that $-f$ is continuous.

Let us consider T and let f be a non-empty continuous function from T into \mathbb{R}^1 . One can verify that f^{-1} is continuous.

Let us consider T , let f be a continuous function from T into \mathbb{R}^1 , and let us consider r . One can check the following observations:

- * $f + r$ is continuous,
- * $f - r$ is continuous,
- * $f r$ is continuous, and
- * f/r is continuous.

Let us consider T and let f, g be continuous functions from T into \mathbb{R}^1 . One can verify the following observations:

- * $f + g$ is continuous,
- * $f - g$ is continuous, and
- * $f g$ is continuous.

Let us consider T , let f be a continuous function from T into \mathbb{R}^1 , and let g be a non-empty continuous function from T into \mathbb{R}^1 . Observe that f/g is continuous.

Let us consider n, T and let f, g be continuous functions from T into \mathcal{E}_T^n . One can verify the following observations:

- * $f + g$ is continuous,
- * $f - g$ is continuous, and
- * $f \cdot g$ is continuous.

Let us consider n, T , let f be a continuous function from T into \mathcal{E}_T^n , and let g be a continuous function from T into \mathbb{R}^1 . One can verify the following observations:

- * $f + g$ is continuous,
- * $f - g$ is continuous, and
- * $f \cdot g$ is continuous.

Let us consider n, T , let f be a continuous function from T into \mathcal{E}_T^n , and let g be a non-empty continuous function from T into \mathbb{R}^1 . Observe that f/g is continuous.

Let us consider n, T, r and let f be a continuous function from T into \mathcal{E}_T^n . One can verify the following observations:

- * $f + r$ is continuous,
- * $f - r$ is continuous,
- * $f \cdot r$ is continuous, and
- * f/r is continuous.

We now state two propositions:

- (60) Let r be a non negative real number, n be a non zero natural number, and p be a point of $\text{Tcircle}(0_{\mathcal{E}_T^n}, r)$. Then $-p$ is a point of $\text{Tcircle}(0_{\mathcal{E}_T^n}, r)$.

- (61) Let r be a non negative real number and f be a function from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, r)$ into \mathcal{E}_T^n . Then $f \circ -$ is a function from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, r)$ into \mathcal{E}_T^n .

Let n be a natural number, let r be a non negative real number, and let X be a subset of $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, r)$. Then $(-)X$ is a subset of $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, r)$.

Let us consider m , let r be a non negative real number, and let X be an open subset of $\text{Tcircle}(0_{\mathcal{E}_T^{m+1}}, r)$. One can verify that $(-)X$ is open.

The following proposition is true

- (62) Let r be a non negative real number and f be a continuous function from $\text{Tcircle}(0_{\mathcal{E}_T^{m+1}}, r)$ into \mathcal{E}_T^m . Then $f \circ -$ is a continuous function from $\text{Tcircle}(0_{\mathcal{E}_T^{m+1}}, r)$ into \mathcal{E}_T^m .

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