

# The Geometric Interior in Real Linear Spaces

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**Summary.** We introduce the notions of the geometric interior and the centre of mass for subsets of real linear spaces. We prove a number of theorems concerning these notions which are used in the theory of abstract simplicial complexes.

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The papers [1], [6], [11], [2], [5], [3], [4], [13], [7], [16], [10], [14], [12], [8], [9], and [15] provide the terminology and notation for this paper.

## 1. PRELIMINARIES

For simplicity, we adopt the following convention:  $x$  denotes a set,  $r, s$  denote real numbers,  $n$  denotes a natural number,  $V$  denotes a real linear space,  $v, u, w, p$  denote vectors of  $V$ ,  $A, B$  denote subsets of  $V$ ,  $A_1$  denotes a finite subset of  $V$ ,  $I$  denotes an affinely independent subset of  $V$ ,  $I_1$  denotes a finite affinely independent subset of  $V$ ,  $F$  denotes a family of subsets of  $V$ , and  $L_1, L_2$  denote linear combinations of  $V$ .

Next we state four propositions:

- (1) Let  $L$  be a linear combination of  $A$ . Suppose  $L$  is convex and  $v \neq \sum L$  and  $L(v) \neq 0$ . Then there exists  $p$  such that  $p \in \text{conv } A \setminus \{v\}$  and  $\sum L = L(v) \cdot v + (1 - L(v)) \cdot p$  and  $\frac{1}{L(v)} \cdot \sum L + (1 - \frac{1}{L(v)}) \cdot p = v$ .
- (2) Let  $p_1, p_2, w_1, w_2$  be elements of  $V$ . Suppose that  $v, u \in \text{conv } I$  and  $u \notin \text{conv } I \setminus \{p_1\}$  and  $u \notin \text{conv } I \setminus \{p_2\}$  and  $w_1 \in \text{conv } I \setminus \{p_1\}$  and

$w_2 \in \text{conv } I \setminus \{p_2\}$  and  $r \cdot u + (1 - r) \cdot w_1 = v$  and  $s \cdot u + (1 - s) \cdot w_2 = v$  and  $r < 1$  and  $s < 1$ . Then  $w_1 = w_2$  and  $r = s$ .

- (3) Let  $L$  be a linear combination of  $A_1$ . Suppose  $A_1 \subseteq \text{conv } I_1$  and  $\text{sum } L = 1$ . Then
- (i)  $\sum L \in \text{Affin } I_1$ , and
  - (ii) for every element  $x$  of  $V$  there exists a finite sequence  $F$  of elements of  $\mathbb{R}$  and there exists a finite sequence  $G$  of elements of  $V$  such that  $(\sum L \rightarrow I_1)(x) = \sum F$  and  $\text{len } G = \text{len } F$  and  $G$  is one-to-one and  $\text{rng } G = \text{the support of } L$  and for every  $n$  such that  $n \in \text{dom } F$  holds  $F(n) = L(G(n)) \cdot (G(n) \rightarrow I_1)(x)$ .
- (4) For every subset  $A_2$  of  $V$  such that  $A_2$  is affine and  $\text{conv } A \cap \text{conv } B \subseteq A_2$  and  $\text{conv } A \setminus \{v\} \subseteq A_2$  and  $v \notin A_2$  holds  $\text{conv } A \setminus \{v\} \cap \text{conv } B = \text{conv } A \cap \text{conv } B$ .

## 2. THE GEOMETRIC INTERIOR

Let  $V$  be a non empty RLS structure and let  $A$  be a subset of  $V$ . The functor  $\text{Int } A$  yields a subset of  $V$  and is defined by:

- (Def. 1)  $x \in \text{Int } A$  iff  $x \in \text{conv } A$  and it is not true that there exists a subset  $B$  of  $V$  such that  $B \subset A$  and  $x \in \text{conv } B$ .

Let  $V$  be a non empty RLS structure and let  $A$  be an empty subset of  $V$ . Observe that  $\text{Int } A$  is empty.

We now state a number of propositions:

- (5) For every non empty RLS structure  $V$  and for every subset  $A$  of  $V$  holds  $\text{Int } A \subseteq \text{conv } A$ .
- (6) Let  $V$  be a real linear space-like non empty RLS structure and  $A$  be a subset of  $V$ . Then  $\text{Int } A = A$  if and only if  $A$  is trivial.
- (7) If  $A \subset B$ , then  $\text{conv } A$  misses  $\text{Int } B$ .
- (8)  $\text{conv } A = \bigcup \{\text{Int } B : B \subseteq A\}$ .
- (9)  $\text{conv } A = \text{Int } A \cup \bigcup \{\text{conv } A \setminus \{v\} : v \in A\}$ .
- (10) If  $x \in \text{Int } A$ , then there exists a linear combination  $L$  of  $A$  such that  $L$  is convex and  $x = \sum L$ .
- (11) For every linear combination  $L$  of  $A$  such that  $L$  is convex and  $\sum L \in \text{Int } A$  holds the support of  $L = A$ .
- (12) For every linear combination  $L$  of  $I$  such that  $L$  is convex and the support of  $L = I$  holds  $\sum L \in \text{Int } I$ .
- (13) If  $\text{Int } A$  is non empty, then  $A$  is finite.
- (14) If  $v \in I$  and  $u \in \text{Int } I$  and  $p \in \text{conv } I \setminus \{v\}$  and  $r \cdot v + (1 - r) \cdot p = u$ , then  $p \in \text{Int}(I \setminus \{v\})$ .

3. THE CENTER OF MASS

Let us consider  $V$ . The center of mass of  $V$  yielding a function from  $2_+^{\text{the carrier of } V}$  into the carrier of  $V$  is defined by the conditions (Def. 2).

- (Def. 2)(i) For every non empty finite subset  $A$  of  $V$  holds (the center of mass of  $V$ )( $A$ ) =  $\frac{1}{\#A} \cdot \sum A$ , and
- (ii) for every  $A$  such that  $A$  is infinite holds (the center of mass of  $V$ )( $A$ ) =  $0_V$ .

One can prove the following propositions:

- (15) There exists a linear combination  $L$  of  $A_1$  such that  $\sum L = r \cdot \sum A_1$  and  $\text{sum } L = r \cdot \overline{\overline{A_1}}$  and  $L = \mathbf{0}_{LCV} + \cdot (A_1 \mapsto r)$ .
- (16) If  $A_1$  is non empty, then (the center of mass of  $V$ )( $A_1$ )  $\in \text{conv } A_1$ .
- (17) If  $\bigcup F$  is finite, then (the center of mass of  $V$ ) $^\circ F \subseteq \text{conv } \bigcup F$ .
- (18) If  $v \in I_1$ , then ((the center of mass of  $V$ )( $I_1$ )  $\rightarrow I_1$ )( $v$ ) =  $\frac{1}{\#I_1}$ .
- (19) (The center of mass of  $V$ )( $I_1$ )  $\in I_1$  iff  $\overline{\overline{I_1}} = 1$ .
- (20) If  $I_1$  is non empty, then (the center of mass of  $V$ )( $I_1$ )  $\in \text{Int } I_1$ .
- (21) If  $A \subseteq I_1$  and (the center of mass of  $V$ )( $I_1$ )  $\in \text{Affin } A$ , then  $I_1 = A$ .
- (22) If  $v \in A_1$  and  $A_1 \setminus \{v\}$  is non empty, then (the center of mass of  $V$ )( $A_1$ ) =  $(1 - \frac{1}{\#A_1}) \cdot (\text{the center of mass of } V)_{A_1 \setminus \{v\}} + \frac{1}{\#A_1} \cdot v$ .
- (23) If  $\text{conv } A \subseteq \text{conv } I_1$  and  $I_1$  is non empty and  $\text{conv } A$  misses  $\text{Int } I_1$ , then there exists a subset  $B$  of  $V$  such that  $B \subset I_1$  and  $\text{conv } A \subseteq \text{conv } B$ .
- (24) If  $\sum L_1 \neq \sum L_2$  and  $\text{sum } L_1 = \text{sum } L_2$ , then there exists  $v$  such that  $L_1(v) > L_2(v)$ .
- (25) Let  $p$  be a real number. Suppose  $(r \cdot L_1 + (1 - r) \cdot L_2)(v) \leq p \leq (s \cdot L_1 + (1 - s) \cdot L_2)(v)$ . Then there exists a real number  $r_1$  such that  $(r_1 \cdot L_1 + (1 - r_1) \cdot L_2)(v) = p$  and if  $r \leq s$ , then  $r \leq r_1 \leq s$  and if  $s \leq r$ , then  $s \leq r_1 \leq r$ .
- (26) If  $v, u \in \text{conv } A$  and  $v \neq u$ , then there exist  $p, w, r$  such that  $p \in A$  and  $w \in \text{conv } A \setminus \{p\}$  and  $0 \leq r < 1$  and  $r \cdot u + (1 - r) \cdot w = v$ .
- (27)  $A \cup \{v\}$  is affinely independent iff  $A$  is affinely independent but  $v \in A$  or  $v \notin \text{Affin } A$ .
- (28) If  $A_1 \subseteq I$  and  $v \in A_1$ , then  $(I \setminus \{v\}) \cup \{(\text{the center of mass of } V)(A_1)\}$  is an affinely independent subset of  $V$ .
- (29) Let  $F$  be a  $\subseteq$ -linear family of subsets of  $V$ . Suppose  $\bigcup F$  is finite and affinely independent. Then (the center of mass of  $V$ ) $^\circ F$  is an affinely independent subset of  $V$ .
- (30) Let  $F$  be a  $\subseteq$ -linear family of subsets of  $V$ . Suppose  $\bigcup F$  is affinely independent and finite. Then  $\text{Int}((\text{the center of mass of } V)^\circ F) \subseteq \text{Int } \bigcup F$ .

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