

# Sperner's Lemma

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**Summary.** In this article we introduce and prove properties of simplicial complexes in real linear spaces which are necessary to formulate Sperner's lemma. The lemma states that for a function  $f$ , which for an arbitrary vertex  $v$  of the barycentric subdivision  $\mathcal{B}$  of simplex  $\mathcal{K}$  assigns some vertex from a face of  $\mathcal{K}$  which contains  $v$ , we can find a simplex  $S$  of  $\mathcal{B}$  which satisfies  $f(S) = \mathcal{K}$  (see [10]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [11], [19], [9], [6], [7], [1], [5], [3], [4], [13], [15], [12], [22], [23], [16], [18], [20], [14], [17], [21], and [8].

## 1. PRELIMINARIES

We follow the rules:  $x, y, X$  denote sets and  $n, k$  denote natural numbers.

The following two propositions are true:

- (1) Let  $R$  be a binary relation and  $C$  be a cardinal number. If for every  $x$  such that  $x \in X$  holds  $\text{Card}(R^\circ x) = C$ , then  $\text{Card } R = \text{Card}(R \upharpoonright (\text{dom } R \setminus X)) + C \cdot \text{Card } X$ .
- (2) Let  $Y$  be a non empty finite set. Suppose  $\text{Card } X = \overline{\overline{Y}} + 1$ . Let  $f$  be a function from  $X$  into  $Y$ . Suppose  $f$  is onto. Then there exists  $y$  such that  $y \in Y$  and  $\text{Card}(f^{-1}(\{y\})) = 2$  and for every  $x$  such that  $x \in Y$  and  $x \neq y$  holds  $\text{Card}(f^{-1}(\{x\})) = 1$ .

Let  $X$  be a 1-sorted structure. A simplicial complex structure of  $X$  is a simplicial complex structure of the carrier of  $X$ . A simplicial complex of  $X$  is a simplicial complex of the carrier of  $X$ .

Let  $X$  be a 1-sorted structure, let  $K$  be a simplicial complex structure of  $X$ , and let  $A$  be a subset of  $K$ . The functor  ${}^{\textcircled{A}}$  yielding a subset of  $X$  is defined by:

(Def. 1)  ${}^{\textcircled{A}}A = A$ .

Let  $X$  be a 1-sorted structure, let  $K$  be a simplicial complex structure of  $X$ , and let  $A$  be a family of subsets of  $K$ . The functor  ${}^{\textcircled{A}}$  yielding a family of subsets of  $X$  is defined by:

(Def. 2)  ${}^{\textcircled{A}}A = A$ .

We now state the proposition

- (3) Let  $X$  be a 1-sorted structure and  $K$  be a subset-closed simplicial complex structure of  $X$ . Suppose  $K$  is total. Let  $S$  be a finite subset of  $K$ . Suppose  $S$  is simplex-like. Then the complex of  $\{^{\textcircled{S}}\}$  is a subsimplicial complex of  $K$ .

## 2. THE AREA OF AN ABSTRACT SIMPLICIAL COMPLEX

For simplicity, we adopt the following rules:  $R_1$  denotes a non empty RLS structure,  $K_1, K_2, K_3$  denote simplicial complex structures of  $R_1$ ,  $V$  denotes a real linear space, and  $K_4$  denotes a non void simplicial complex of  $V$ .

Let us consider  $R_1, K_1$ . The functor  $|K_1|$  yields a subset of  $R_1$  and is defined by:

(Def. 3)  $x \in |K_1|$  iff there exists a subset  $A$  of  $K_1$  such that  $A$  is simplex-like and  $x \in \text{conv}^{\textcircled{A}}$ .

One can prove the following propositions:

- (4) If the topology of  $K_2 \subseteq$  the topology of  $K_3$ , then  $|K_2| \subseteq |K_3|$ .
- (5) For every subset  $A$  of  $K_1$  such that  $A$  is simplex-like holds  $\text{conv}^{\textcircled{A}} \subseteq |K_1|$ .
- (6) Let  $K$  be a subset-closed simplicial complex structure of  $V$ . Then  $x \in |K|$  if and only if there exists a subset  $A$  of  $K$  such that  $A$  is simplex-like and  $x \in \text{Int}^{\textcircled{A}}$ .
- (7)  $|K_1|$  is empty iff  $K_1$  is empty-membered.
- (8) For every subset  $A$  of  $R_1$  holds  $|\text{the complex of } \{A\}| = \text{conv } A$ .
- (9) For all families  $A, B$  of subsets of  $R_1$  holds  $|\text{the complex of } A \cup B| = |\text{the complex of } A| \cup |\text{the complex of } B|$ .

## 3. THE SUBDIVISION OF A SIMPLICIAL COMPLEX

Let us consider  $R_1, K_1$ . A simplicial complex structure of  $R_1$  is said to be a subdivision structure of  $K_1$  if it satisfies the conditions (Def. 4).

- (Def. 4)(i)  $|K_1| \subseteq |it|$ , and  
(ii) for every subset  $A$  of it such that  $A$  is simplex-like there exists a subset  $B$  of  $K_1$  such that  $B$  is simplex-like and  $\text{conv}^{\textcircled{a}} A \subseteq \text{conv}^{\textcircled{a}} B$ .

The following proposition is true

- (10) For every subdivision structure  $P$  of  $K_1$  holds  $|K_1| = |P|$ .

Let us consider  $R_1$  and let  $K_1$  be a simplicial complex structure of  $R_1$  with a non-empty element. Observe that every subdivision structure of  $K_1$  has a non-empty element.

We now state four propositions:

- (11)  $K_1$  is a subdivision structure of  $K_1$ .  
(12) The complex of the topology of  $K_1$  is a subdivision structure of  $K_1$ .  
(13) Let  $K$  be a subset-closed simplicial complex structure of  $V$  and  $S_1$  be a family of subsets of  $K$ . Suppose  $S_1 = \text{SubFin}(\text{the topology of } K)$ . Then the complex of  $S_1$  is a subdivision structure of  $K$ .  
(14) For every subdivision structure  $P_1$  of  $K_1$  holds every subdivision structure of  $P_1$  is a subdivision structure of  $K_1$ .

Let us consider  $V$  and let  $K$  be a simplicial complex structure of  $V$ . Note that there exists a subdivision structure of  $K$  which is finite-membered and subset-closed.

Let us consider  $V$  and let  $K$  be a simplicial complex structure of  $V$ . A subdivision of  $K$  is a finite-membered subset-closed subdivision structure of  $K$ .

We now state the proposition

- (15) Let  $K$  be a simplicial complex of  $V$  with empty element. Suppose  $|K| \subseteq \Omega_K$ . Let  $B$  be a function from  $2_+^{\text{the carrier of } V}$  into the carrier of  $V$ . Suppose that for every simplex  $S$  of  $K$  such that  $S$  is non empty holds  $B(S) \in \text{conv}^{\textcircled{a}} S$ . Then  $\text{subdivision}(B, K)$  is a subdivision structure of  $K$ .

Let us consider  $V, K_4$ . One can verify that there exists a subdivision of  $K_4$  which is non void.

## 4. THE BARYCENTRIC SUBDIVISION

Let us consider  $V, K_4$ . Let us assume that  $|K_4| \subseteq \Omega_{(K_4)}$ . The functor  $\text{BCS } K_4$  yields a non void subdivision of  $K_4$  and is defined by:

- (Def. 5)  $\text{BCS } K_4 = \text{subdivision}(\text{the center of mass of } V, K_4)$ .

Let us consider  $n$  and let us consider  $V, K_4$ . Let us assume that  $|K_4| \subseteq \Omega_{(K_4)}$ . The functor  $\text{BCS}(n, K_4)$  yields a non void subdivision of  $K_4$  and is defined by:

(Def. 6)  $\text{BCS}(n, K_4) = \text{subdivision}(n, \text{the center of mass of } V, K_4)$ .

Next we state several propositions:

- (16) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $\text{BCS}(0, K_4) = K_4$ .
- (17) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $\text{BCS}(1, K_4) = \text{BCS } K_4$ .
- (18) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $\Omega_{\text{BCS}(n, K_4)} = \Omega_{(K_4)}$ .
- (19) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $|\text{BCS}(n, K_4)| = |K_4|$ .
- (20) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $\text{BCS}(n+1, K_4) = \text{BCS } \text{BCS}(n, K_4)$ .
- (21) If  $|K_4| \subseteq \Omega_{(K_4)}$  and  $\text{degree}(K_4) \leq 0$ , then the topological structure of  $K_4 = \text{BCS } K_4$ .
- (22) If  $n > 0$  and  $|K_4| \subseteq \Omega_{(K_4)}$  and  $\text{degree}(K_4) \leq 0$ , then the topological structure of  $K_4 = \text{BCS}(n, K_4)$ .
- (23) Let  $S_2$  be a non void subsimplicial complex of  $K_4$ . If  $|K_4| \subseteq \Omega_{(K_4)}$  and  $|S_2| \subseteq \Omega_{(S_2)}$ , then  $\text{BCS}(n, S_2)$  is a subsimplicial complex of  $\text{BCS}(n, K_4)$ .
- (24) If  $|K_4| \subseteq \Omega_{(K_4)}$ , then  $\text{Vertices } K_4 \subseteq \text{Vertices } \text{BCS}(n, K_4)$ .

Let us consider  $n, V$  and let  $K$  be a non void total simplicial complex of  $V$ . Note that  $\text{BCS}(n, K)$  is total.

Let us consider  $n, V$  and let  $K$  be a non void finite-vertices total simplicial complex of  $V$ . Note that  $\text{BCS}(n, K)$  is finite-vertices.

## 5. SELECTED PROPERTIES OF SIMPLICIAL COMPLEXES

Let us consider  $V$  and let  $K$  be a simplicial complex structure of  $V$ . We say that  $K$  is affinely-independent if and only if:

(Def. 7) For every subset  $A$  of  $K$  such that  $A$  is simplex-like holds  ${}^{\textcircled{A}}$   $A$  is affinely-independent.

Let us consider  $R_1, K_1$ . We say that  $K_1$  is simplex-join-closed if and only if:

(Def. 8) For all subsets  $A, B$  of  $K_1$  such that  $A$  is simplex-like and  $B$  is simplex-like holds  $\text{conv}^{\textcircled{A}} A \cap \text{conv}^{\textcircled{B}} B = \text{conv}^{\textcircled{A \cap B}} A \cap B$ .

Let us consider  $V$ . Note that every simplicial complex structure of  $V$  which is empty-membered is also affinely-independent. Let  $F$  be an affinely-independent family of subsets of  $V$ . Observe that the complex of  $F$  is affinely-independent.

Let us consider  $R_1$ . One can verify that every simplicial complex structure of  $R_1$  which is empty-membered is also simplex-join-closed.

Let us consider  $V$  and let  $I$  be an affinely-independent subset of  $V$ . One can check that the complex of  $\{I\}$  is simplex-join-closed.

Let us consider  $V$ . One can check that there exists a subset of  $V$  which is non empty, trivial, and affinely-independent.

Let us consider  $V$ . One can check that there exists a simplicial complex of  $V$  which is finite-vertices, affinely-independent, simplex-join-closed, and total and has a non-empty element.

Let us consider  $V$  and let  $K$  be an affinely-independent simplicial complex structure of  $V$ . One can verify that every subsimplicial complex of  $K$  is affinely-independent.

Let us consider  $V$  and let  $K$  be a simplex-join-closed simplicial complex structure of  $V$ . One can check that every subsimplicial complex of  $K$  is simplex-join-closed.

Next we state the proposition

- (25) Let  $K$  be a subset-closed simplicial complex structure of  $V$ . Then  $K$  is simplex-join-closed if and only if for all subsets  $A, B$  of  $K$  such that  $A$  is simplex-like and  $B$  is simplex-like and  $\text{Int}(@A)$  meets  $\text{Int}(@B)$  holds  $A = B$ .

For simplicity, we follow the rules:  $K_5$  is a simplex-join-closed simplicial complex of  $V$ ,  $A_1, B_1$  are subsets of  $K_5$ ,  $K_6$  is a non void affinely-independent simplicial complex of  $V$ ,  $K_7$  is a non void affinely-independent simplex-join-closed simplicial complex of  $V$ , and  $K$  is a non void affinely-independent simplex-join-closed total simplicial complex of  $V$ .

Let us consider  $V, K_6$  and let  $S$  be a simplex of  $K_6$ . Note that  $@S$  is affinely-independent.

One can prove the following propositions:

- (26) If  $A_1$  is simplex-like and  $B_1$  is simplex-like and  $\text{Int}(@A_1)$  meets  $\text{conv}^@B_1$ , then  $A_1 \subseteq B_1$ .
- (27) If  $A_1$  is simplex-like and  $@A_1$  is affinely-independent and  $B_1$  is simplex-like, then  $\text{Int}(@A_1) \subseteq \text{conv}^@B_1$  iff  $A_1 \subseteq B_1$ .
- (28) If  $|K_6| \subseteq \Omega_{(K_6)}$ , then  $\text{BCS } K_6$  is affinely-independent.

Let us consider  $V$  and let  $K_6$  be a non void affinely-independent total simplicial complex of  $V$ . Observe that  $\text{BCS } K_6$  is affinely-independent. Let us consider  $n$ . Observe that  $\text{BCS}(n, K_6)$  is affinely-independent.

Let us consider  $V, K_7$ . One can verify that (the center of mass of  $V$ )|the topology of  $K_7$  is one-to-one.

We now state the proposition

- (29) If  $|K_7| \subseteq \Omega_{(K_7)}$ , then  $\text{BCS } K_7$  is simplex-join-closed.

Let us consider  $V, K$ . Note that  $\text{BCS } K$  is simplex-join-closed. Let us consider  $n$ . Observe that  $\text{BCS}(n, K)$  is simplex-join-closed.

The following four propositions are true:

- (30) Suppose  $|K_4| \subseteq \Omega_{(K_4)}$  and for every  $n$  such that  $n \leq \text{degree}(K_4)$  there exists a simplex  $S$  of  $K_4$  such that  $\overline{S} = n + 1$  and  $@S$  is affinely-independent. Then  $\text{degree}(K_4) = \text{degree}(\text{BCS } K_4)$ .
- (31) If  $|K_6| \subseteq \Omega_{(K_6)}$ , then  $\text{degree}(K_6) = \text{degree}(\text{BCS } K_6)$ .
- (32) If  $|K_6| \subseteq \Omega_{(K_6)}$ , then  $\text{degree}(K_6) = \text{degree}(\text{BCS}(n, K_6))$ .

- (33) Let  $S$  be a simplex-like family of subsets of  $K_7$ . If  $S$  has non empty elements, then  $\text{Card } S = \text{Card}((\text{the center of mass of } V)^\circ S)$ .

For simplicity, we adopt the following convention:  $A_2$  denotes a finite affinely-independent subset of  $V$ ,  $A_3, B_2$  denote finite subsets of  $V$ ,  $B$  denotes a subset of  $V$ ,  $S, T$  denote finite families of subsets of  $V$ ,  $S_3$  denotes a  $\subseteq$ -linear finite finite-membered family of subsets of  $V$ ,  $S_4, T_1$  denote finite simplex-like families of subsets of  $K$ , and  $A_4$  denotes a simplex of  $K$ .

The following propositions are true:

- (34) Let  $S_6, S_5$  be simplex-like families of subsets of  $K_7$ . Suppose that
- (i)  $|K_7| \subseteq \Omega_{(K_7)}$ ,
  - (ii)  $S_6$  has non empty elements,
  - (iii)  $(\text{the center of mass of } V)^\circ S_5$  is a simplex of BCS  $K_7$ , and
  - (iv)  $(\text{the center of mass of } V)^\circ S_6 \subseteq (\text{the center of mass of } V)^\circ S_5$ .
- Then  $S_6 \subseteq S_5$  and  $S_5$  is  $\subseteq$ -linear.
- (35) Suppose  $S$  has non empty elements and  $\bigcup S \subseteq A_2$  and  $\overline{\overline{S}} + n + 1 \leq \overline{\overline{A_2}}$ . Then the following statements are equivalent
- (i)  $B_2$  is a simplex of  $n + \overline{\overline{S}}$  and BCS (the complex of  $\{A_2\}$ ) and  $(\text{the center of mass of } V)^\circ S \subseteq B_2$ ,
  - (ii) there exists  $T$  such that  $T$  misses  $S$  and  $T \cup S$  is  $\subseteq$ -linear and has non empty elements and  $\overline{\overline{T}} = n + 1$  and  $\bigcup T \subseteq A_2$  and  $B_2 = (\text{the center of mass of } V)^\circ S \cup (\text{the center of mass of } V)^\circ T$ .
- (36) Suppose  $S_3$  has non empty elements and  $\bigcup S_3 \subseteq A_2$ . Then the following statements are equivalent
- (i)  $(\text{the center of mass of } V)^\circ S_3$  is a simplex of  $\overline{\overline{\bigcup S_3}} - 1$  and BCS (the complex of  $\{A_2\}$ ),
  - (ii) for every  $n$  such that  $0 < n \leq \overline{\overline{\bigcup S_3}}$  there exists  $x$  such that  $x \in S_3$  and  $\text{Card } x = n$ .
- (37) Let given  $S$ . Suppose  $S$  is  $\subseteq$ -linear and has non empty elements and  $\overline{\overline{S}} = \text{Card} \bigcup S$ . Let given  $A_3, B_2$ . Suppose  $A_3$  is non empty and  $A_3$  misses  $\bigcup S$  and  $\bigcup S \cup A_3$  is affinely-independent and  $\bigcup S \cup A_3 \subseteq B_2$ . Then  $(\text{the center of mass of } V)^\circ S \cup (\text{the center of mass of } V)^\circ \{\bigcup S \cup A_3\}$  is a simplex of  $\overline{\overline{S}}$  and BCS (the complex of  $\{B_2\}$ ).
- (38) Let given  $S_3$ . Suppose  $S_3$  has non empty elements and  $\overline{\overline{S_3}} = \overline{\overline{\bigcup S_3}}$ . Let  $v$  be an element of  $V$ . Suppose  $v \notin \bigcup S_3$  and  $\bigcup S_3 \cup \{v\}$  is affinely-independent. Then  $\{S_6; S_6 \text{ ranges over simplexes of } \overline{\overline{S_3}} \text{ and BCS (the complex of } \{\bigcup S_3 \cup \{v\}\})\}$ :  $(\text{the center of mass of } V)^\circ S_3 \subseteq S_6 = \{(\text{the center of mass of } V)^\circ S_3 \cup (\text{the center of mass of } V)^\circ \{\bigcup S_3 \cup \{v\}\})\}$ .
- (39) Let given  $S_3$ . Suppose  $S_3$  has non empty elements and  $\overline{\overline{S_3}} + 1 = \overline{\overline{\bigcup S_3}}$  and  $\bigcup S_3$  is affinely-independent. Then  $\text{Card}\{S_6; S_6 \text{ ranges over simplexes of } \overline{\overline{S_3}} \text{ and BCS (the complex of } \{\bigcup S_3\})\}$ :  $(\text{the center of mass of } V)^\circ S_3 \subseteq S_6 = 2$ .

- (40) Suppose  $A_2$  is a simplex of  $K$ . Then  $B$  is a simplex of BCS (the complex of  $\{A_2\}$ ) if and only if  $B$  is a simplex of BCS  $K$  and  $\text{conv } B \subseteq \text{conv } A_2$ .
- (41) Suppose  $S_4$  has non empty elements and  $\overline{S_4} + n \leq \text{degree}(K)$ . Then the following statements are equivalent
- (i)  $A_3$  is a simplex of  $n + \overline{S_4}$  and BCS  $K$  and (the center of mass of  $V$ ) $^\circ S_4 \subseteq A_3$ ,
  - (ii) there exists  $T_1$  such that  $T_1$  misses  $S_4$  and  $T_1 \cup S_4$  is  $\subseteq$ -linear and has non empty elements and  $\overline{T_1} = n + 1$  and  $A_3 =$  (the center of mass of  $V$ ) $^\circ S_4 \cup$  (the center of mass of  $V$ ) $^\circ T_1$ .
- (42) Suppose  $S_4$  is  $\subseteq$ -linear and has non empty elements and  $\overline{S_4} = \overline{\bigcup S_4}$  and  $\bigcup S_4 \subseteq A_4$  and  $\overline{A_4} = \overline{S_4} + 1$ . Then  $\{S_6; S_6 \text{ ranges over simplexes of } \overline{S_4} \text{ and BCS } K : (\text{the center of mass of } V)^\circ S_4 \subseteq S_6 \wedge \text{conv}^\circ S_6 \subseteq \text{conv}^\circ A_4\} = \{(\text{the center of mass of } V)^\circ S_4 \cup (\text{the center of mass of } V)^\circ \{A_4\}\}$ .
- (43) Suppose  $S_4$  is  $\subseteq$ -linear and has non empty elements and  $\overline{S_4} + 1 = \overline{\bigcup S_4}$ . Then  $\text{Card}\{S_6; S_6 \text{ ranges over simplexes of } \overline{S_4} \text{ and BCS } K : (\text{the center of mass of } V)^\circ S_4 \subseteq S_6 \wedge \text{conv}^\circ S_6 \subseteq \text{conv}^\circ \bigcup S_4\} = 2$ .
- (44) Let given  $A_3$ . Suppose that
- (i)  $K$  is a subdivision of the complex of  $\{A_3\}$ ,
  - (ii)  $\overline{A_3} = n + 1$ ,
  - (iii)  $\text{degree}(K) = n$ , and
  - (iv) for every simplex  $S$  of  $n - 1$  and  $K$  and for every  $X$  such that  $X = \{S_6; S_6 \text{ ranges over simplexes of } n \text{ and } K : S \subseteq S_6\}$  holds if  $\text{conv}^\circ S$  meets  $\text{Int } A_3$ , then  $\text{Card } X = 2$  and if  $\text{conv}^\circ S$  misses  $\text{Int } A_3$ , then  $\text{Card } X = 1$ . Let  $S$  be a simplex of  $n - 1$  and BCS  $K$  and given  $X$  such that  $X = \{S_6; S_6 \text{ ranges over simplexes of } n \text{ and BCS } K : S \subseteq S_6\}$ . Then
  - (v) if  $\text{conv}^\circ S$  meets  $\text{Int } A_3$ , then  $\text{Card } X = 2$ , and
  - (vi) if  $\text{conv}^\circ S$  misses  $\text{Int } A_3$ , then  $\text{Card } X = 1$ .
- (45) Let  $S$  be a simplex of  $n - 1$  and BCS( $k$ , the complex of  $\{A_2\}$ ) such that  $\overline{A_2} = n + 1$  and  $X = \{S_6; S_6 \text{ ranges over simplexes of } n \text{ and BCS}(k, \text{ the complex of } \{A_2\}): S \subseteq S_6\}$ . Then
- (i) if  $\text{conv}^\circ S$  meets  $\text{Int } A_2$ , then  $\text{Card } X = 2$ , and
  - (ii) if  $\text{conv}^\circ S$  misses  $\text{Int } A_2$ , then  $\text{Card } X = 1$ .

## 6. THE MAIN THEOREM

In the sequel  $v$  is a vertex of BCS( $k$ , the complex of  $\{A_2\}$ ) and  $F$  is a function from Vertices BCS( $k$ , the complex of  $\{A_2\}$ ) into  $A_2$ .

The following two propositions are true:

- (46) Let given  $F$ . Suppose that for all  $v, B$  such that  $B \subseteq A_2$  and  $v \in \text{conv } B$  holds  $F(v) \in B$ . Then there exists  $n$  such that  $\text{Card}\{S; S \text{ ranges over}$

simplexes of  $\overline{A_2} - 1$  and  $\text{BCS}(k, \text{the complex of } \{A_2\})$ :  $F^\circ S = A_2\} = 2 \cdot n + 1$ .

- (47) Let given  $F$ . Suppose that for all  $v, B$  such that  $B \subseteq A_2$  and  $v \in \text{conv } B$  holds  $F(v) \in B$ . Then there exists a simplex  $S$  of  $\overline{A_2} - 1$  and  $\text{BCS}(k, \text{the complex of } \{A_2\})$  such that  $F^\circ S = A_2$ .

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