FORMALIZED MATHEMATICS
Vol. 18, No. 4, Pages 189–196, 2010
DOI: 10.2478/v10037-010-0022-x

Sperner's Lemma

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. In this article we introduce and prove properties of simplicial complexes in real linear spaces which are necessary to formulate Sperner's lemma. The lemma states that for a function f, which for an arbitrary vertex v of the barycentric subdivision $\mathcal{B}$ of simplex $\mathcal{K}$ assigns some vertex from a face of $\mathcal{K}$ which contains v, we can find a simplex S of $\mathcal{B}$ which satisfies $f(S) = \mathcal{K}$ (see [10]).

MML identifier: `SIMPLEX1`, version: `7.11.07 4.146.1112`

The notation and terminology used in this paper have been introduced in the following papers: [2], [11], [19], [9], [6], [7], [1], [5], [3], [4], [13], [15], [12], [22], [23], [16], [18], [20], [14], [17], [21], and [8].

1. Preliminaries

We follow the rules: x, y, X denote sets and n, k denote natural numbers.

The following two propositions are true:

(1) Let R be a binary relation and C be a cardinal number. If for every x such that $x \in X$ holds $\mathrm{Card}(R^{\circ}x) = C$, then $\mathrm{Card}\,R = \mathrm{Card}(R{\restriction}(\mathrm{dom}\,R \setminus X)) + C \cdot \mathrm{Card}\,X$.

(2) Let Y be a non empty finite set. Suppose $\mathrm{Card}\,X = \overline{\overline{Y}} + 1$. Let f be a function from X into Y. Suppose f is onto. Then there exists y such that $y \in Y$ and $\mathrm{Card}(f^{-1}(\{y\})) = 2$ and for every x such that $x \in Y$ and $x \neq y$ holds $\mathrm{Card}(f^{-1}(\{x\})) = 1$.

ISSN 1426–2630(p), 1898-9934(e)

Let X be a 1-sorted structure. A simplicial complex structure of X is a simplicial complex structure of the carrier of X. A simplicial complex of X is a simplicial complex of the carrier of X.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a subset of K. The functor ${}^{@}A$ yielding a subset of X is defined by:

(Def. 1) ${}^{@}A = A$.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a family of subsets of K. The functor ${}^{@}A$ yielding a family of subsets of X is defined by:

(Def. 2) ${}^{@}A = A$.

We now state the proposition

(3) Let X be a 1-sorted structure and K be a subset-closed simplicial complex structure of X. Suppose K is total. Let S be a finite subset of K. Suppose S is simplex-like. Then the complex of $\{{}^{@}S\}$ is a subsimplicial complex of K.

2. The Area of an Abstract Simplicial Complex

For simplicity, we adopt the following rules: R_1 denotes a non empty RLS structure, K_1, K_2, K_3 denote simplicial complex structures of R_1, V denotes a real linear space, and K_4 denotes a non void simplicial complex of V.

Let us consider R_1, K_1. The functor $|K_1|$ yields a subset of R_1 and is defined by:

(Def. 3) $x \in |K_1|$ iff there exists a subset A of K_1 such that A is simplex-like and $x \in \operatorname{conv}{}^{@}A$.

One can prove the following propositions:

(4) If the topology of $K_2 \subseteq$ the topology of K_3, then $|K_2| \subseteq |K_3|$.

(5) For every subset A of K_1 such that A is simplex-like holds $\operatorname{conv}{}^{@}A \subseteq |K_1|$.

(6) Let K be a subset-closed simplicial complex structure of V. Then $x \in |K|$ if and only if there exists a subset A of K such that A is simplex-like and $x \in \operatorname{Int}({}^{@}A)$.

(7) $|K_1|$ is empty iff K_1 is empty-membered.

(8) For every subset A of R_1 holds $|$the complex of $\{A\}| = \operatorname{conv} A$.

(9) For all families A, B of subsets of R_1 holds $|$the complex of $A \cup B| = |$the complex of $A| \cup |$the complex of $B|$.

3. The Subdivision of a Simplicial Complex

Let us consider R_1, K_1. A simplicial complex structure of R_1 is said to be a subdivision structure of K_1 if it satisfies the conditions (Def. 4).

(Def. 4)(i) $|K_1| \subseteq |\mathrm{it}|$, and

(ii) for every subset A of it such that A is simplex-like there exists a subset B of K_1 such that B is simplex-like and $\mathrm{conv}^{@}A \subseteq \mathrm{conv}^{@}B$.

The following proposition is true

(10) For every subdivision structure P of K_1 holds $|K_1| = |P|$.

Let us consider R_1 and let K_1 be a simplicial complex structure of R_1 with a non-empty element. Observe that every subdivision structure of K_1 has a non-empty element.

We now state four propositions:

(11) K_1 is a subdivision structure of K_1.

(12) The complex of the topology of K_1 is a subdivision structure of K_1.

(13) Let K be a subset-closed simplicial complex structure of V and S_1 be a family of subsets of K. Suppose $S_1 = \mathrm{SubFin}(\text{the topology of } K)$. Then the complex of S_1 is a subdivision structure of K.

(14) For every subdivision structure P_1 of K_1 holds every subdivision structure of P_1 is a subdivision structure of K_1.

Let us consider V and let K be a simplicial complex structure of V. Note that there exists a subdivision structure of K which is finite-membered and subset-closed.

Let us consider V and let K be a simplicial complex structure of V. A subdivision of K is a finite-membered subset-closed subdivision structure of K.

We now state the proposition

(15) Let K be a simplicial complex of V with empty element. Suppose $|K| \subseteq \Omega_K$. Let B be a function from $2_+^{\text{the carrier of } V}$ into the carrier of V. Suppose that for every simplex S of K such that S is non empty holds $B(S) \in \mathrm{conv}^{@}S$. Then $\mathrm{subdivision}(B, K)$ is a subdivision structure of K.

Let us consider V, K_4. One can verify that there exists a subdivision of K_4 which is non void.

4. The Barycentric Subdivision

Let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}\,K_4$ yields a non void subdivision of K_4 and is defined by:

(Def. 5) $\mathrm{BCS}\,K_4 = \mathrm{subdivision}(\text{the center of mass of } V, K_4)$.

Let us consider n and let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}(n, K_4)$ yields a non void subdivision of K_4 and is defined by:

(Def. 6) $\mathrm{BCS}(n, K_4) = \mathrm{subdivision}(n, \text{the center of mass of } V,\ K_4)$.

Next we state several propositions:

(16) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(0, K_4) = K_4$.

(17) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(1, K_4) = \mathrm{BCS}\, K_4$.

(18) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\Omega_{\mathrm{BCS}(n,K_4)} = \Omega_{(K_4)}$.

(19) If $|K_4| \subseteq \Omega_{(K_4)}$, then $|\mathrm{BCS}(n, K_4)| = |K_4|$.

(20) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(n + 1, K_4) = \mathrm{BCS}\,\mathrm{BCS}(n, K_4)$.

(21) If $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}\, K_4$.

(22) If $n > 0$ and $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}(n, K_4)$.

(23) Let S_2 be a non void subsimplicial complex of K_4. If $|K_4| \subseteq \Omega_{(K_4)}$ and $|S_2| \subseteq \Omega_{(S_2)}$, then $\mathrm{BCS}(n, S_2)$ is a subsimplicial complex of $\mathrm{BCS}(n, K_4)$.

(24) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{Vertices}\, K_4 \subseteq \mathrm{Vertices}\, \mathrm{BCS}(n, K_4)$.

Let us consider n, V and let K be a non void total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is total.

Let us consider n, V and let K be a non void finite-vertices total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is finite-vertices.

5. Selected Properties of Simplicial Complexes

Let us consider V and let K be a simplicial complex structure of V. We say that K is affinely-independent if and only if:

(Def. 7) For every subset A of K such that A is simplex-like holds ${}^{@}A$ is affinely-independent.

Let us consider R_1, K_1. We say that K_1 is simplex-join-closed if and only if:

(Def. 8) For all subsets A, B of K_1 such that A is simplex-like and B is simplex-like holds $\mathrm{conv}^{@}A \cap \mathrm{conv}^{@}B = \mathrm{conv}^{@}A \cap B$.

Let us consider V. Note that every simplicial complex structure of V which is empty-membered is also affinely-independent. Let F be an affinely-independent family of subsets of V. Observe that the complex of F is affinely-independent.

Let us consider R_1. One can verify that every simplicial complex structure of R_1 which is empty-membered is also simplex-join-closed.

Let us consider V and let I be an affinely-independent subset of V. One can check that the complex of $\{I\}$ is simplex-join-closed.

Let us consider V. One can check that there exists a subset of V which is non empty, trivial, and affinely-independent.

Let us consider V. One can check that there exists a simplicial complex of V which is finite-vertices, affinely-independent, simplex-join-closed, and total and has a non-empty element.

Let us consider V and let K be an affinely-independent simplicial complex structure of V. One can verify that every subsimplicial complex of K is affinely-independent.

Let us consider V and let K be a simplex-join-closed simplicial complex structure of V. One can check that every subsimplicial complex of K is simplex-join-closed.

Next we state the proposition

(25) Let K be a subset-closed simplicial complex structure of V. Then K is simplex-join-closed if and only if for all subsets A, B of K such that A is simplex-like and B is simplex-like and $\mathrm{Int}(^{@}A)$ meets $\mathrm{Int}(^{@}B)$ holds $A = B$.

For simplicity, we follow the rules: K_5 is a simplex-join-closed simplicial complex of V, A_1, B_1 are subsets of K_5, K_6 is a non void affinely-independent simplicial complex of V, K_7 is a non void affinely-independent simplex-join-closed simplicial complex of V, and K is a non void affinely-independent simplex-join-closed total simplicial complex of V.

Let us consider V, K_6 and let S be a simplex of K_6. Note that $^{@}S$ is affinely-independent.

One can prove the following propositions:

(26) If A_1 is simplex-like and B_1 is simplex-like and $\mathrm{Int}(^{@}A_1)$ meets $\mathrm{conv}^{@}B_1$, then $A_1 \subseteq B_1$.

(27) If A_1 is simplex-like and $^{@}A_1$ is affinely-independent and B_1 is simplex-like, then $\mathrm{Int}(^{@}A_1) \subseteq \mathrm{conv}^{@}B_1$ iff $A_1 \subseteq B_1$.

(28) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{BCS}\, K_6$ is affinely-independent.

Let us consider V and let K_6 be a non void affinely-independent total simplicial complex of V. Observe that $\mathrm{BCS}\, K_6$ is affinely-independent. Let us consider n. Observe that $\mathrm{BCS}(n, K_6)$ is affinely-independent.

Let us consider V, K_7. One can verify that (the center of mass of V)$\restriction$the topology of K_7 is one-to-one.

We now state the proposition

(29) If $|K_7| \subseteq \Omega_{(K_7)}$, then $\mathrm{BCS}\, K_7$ is simplex-join-closed.

Let us consider V, K. Note that $\mathrm{BCS}\, K$ is simplex-join-closed. Let us consider n. Observe that $\mathrm{BCS}(n, K)$ is simplex-join-closed.

The following four propositions are true:

(30) Suppose $|K_4| \subseteq \Omega_{(K_4)}$ and for every n such that $n \leq \mathrm{degree}(K_4)$ there exists a simplex S of K_4 such that $\overline{\overline{S}} = n + 1$ and $^{@}S$ is affinely-independent. Then $\mathrm{degree}(K_4) = \mathrm{degree}(\mathrm{BCS}\, K_4)$.

(31) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}\, K_6)$.

(32) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}(n, K_6))$.

(33) Let S be a simplex-like family of subsets of K_7. If S has non empty elements, then $\operatorname{Card} S = \operatorname{Card}((\text{the center of mass of } V)^{\circ} S)$.

For simplicity, we adopt the following convention: A_2 denotes a finite affinely-independent subset of V, A_3, B_2 denote finite subsets of V, B denotes a subset of V, S, T denote finite families of subsets of V, S_3 denotes a $\subseteq$-linear finite finite-membered family of subsets of V, S_4, T_1 denote finite simplex-like families of subsets of K, and A_4 denotes a simplex of K.

The following propositions are true:

(34) Let S_6, S_5 be simplex-like families of subsets of K_7. Suppose that
(i) $|K_7| \subseteq \Omega_{(K_7)}$,
(ii) S_6 has non empty elements,
(iii) (the center of mass of $V)^{\circ} S_5$ is a simplex of $\operatorname{BCS} K_7$, and
(iv) (the center of mass of $V)^{\circ} S_6 \subseteq$ (the center of mass of $V)^{\circ} S_5$.

Then $S_6 \subseteq S_5$ and S_5 is $\subseteq$-linear.

(35) Suppose S has non empty elements and $\bigcup S \subseteq A_2$ and $\overline{\overline{S}} + n + 1 \leq \overline{\overline{A_2}}$. Then the following statements are equivalent
(i) B_2 is a simplex of $n + \overline{\overline{S}}$ and BCS (the complex of $\{A_2\}$) and (the center of mass of $V)^{\circ} S \subseteq B_2$,
(ii) there exists T such that T misses S and $T \cup S$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T}} = n + 1$ and $\bigcup T \subseteq A_2$ and $B_2 =$ (the center of mass of $V)^{\circ} S \cup$ (the center of mass of $V)^{\circ} T$.

(36) Suppose S_3 has non empty elements and $\bigcup S_3 \subseteq A_2$. Then the following statements are equivalent
(i) (the center of mass of $V)^{\circ} S_3$ is a simplex of $\overline{\overline{\bigcup S_3}} - 1$ and BCS (the complex of $\{A_2\}$),
(ii) for every n such that $0 < n \leq \overline{\overline{\bigcup S_3}}$ there exists x such that $x \in S_3$ and $\operatorname{Card} x = n$.

(37) Let given S. Suppose S is $\subseteq$-linear and has non empty elements and $\overline{\overline{S}} = \operatorname{Card} \bigcup S$. Let given A_3, B_2. Suppose A_3 is non empty and A_3 misses $\bigcup S$ and $\bigcup S \cup A_3$ is affinely-independent and $\bigcup S \cup A_3 \subseteq B_2$. Then (the center of mass of $V)^{\circ} S \cup$ (the center of mass of $V)^{\circ} \{\bigcup S \cup A_3\}$ is a simplex of $\overline{\overline{S}}$ and BCS (the complex of $\{B_2\}$).

(38) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} = \overline{\overline{\bigcup S_3}}$. Let v be an element of V. Suppose $v \notin \bigcup S_3$ and $\bigcup S_3 \cup \{v\}$ is affinely-independent. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3 \cup \{v\}\}$): (the center of mass of $V)^{\circ} S_3 \subseteq S_6\} = \{$(the center of mass of $V)^{\circ} S_3 \cup$ (the center of mass of $V)^{\circ} \{\bigcup S_3 \cup \{v\}\}\}$.

(39) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} + 1 = \overline{\overline{\bigcup S_3}}$ and $\bigcup S_3$ is affinely-independent. Then $\operatorname{Card}\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3\}$): (the center of mass of $V)^{\circ} S_3 \subseteq S_6\} = 2$.

(40) Suppose A_2 is a simplex of K. Then B is a simplex of BCS (the complex of $\{A_2\}$) if and only if B is a simplex of BCS K and conv $B \subseteq$ conv A_2.

(41) Suppose S_4 has non empty elements and $\overline{\overline{S_4}} + n \leq$ degree(K). Then the following statements are equivalent

(i) A_3 is a simplex of $n + \overline{\overline{S_4}}$ and BCS K and (the center of mass of $V)^{\circ} S_4 \subseteq A_3$,

(ii) there exists T_1 such that T_1 misses S_4 and $T_1 \cup S_4$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T_1}} = n + 1$ and $A_3 =$ (the center of mass of $V)^{\circ} S_4 \cup$ (the center of mass of $V)^{\circ} T_1$.

(42) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} = \overline{\overline{\bigcup S_4}}$ and $\bigcup S_4 \subseteq A_4$ and $\overline{\overline{A_4}} = \overline{\overline{S_4}} + 1$. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^{\circ} S_4 \subseteq S_6 \wedge \mathrm{conv}^{@} S_6 \subseteq \mathrm{conv}^{@} A_4\} =$ $\{$(the center of mass of $V)^{\circ} S_4 \cup$ (the center of mass of $V)^{\circ}\{A_4\}\}$.

(43) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} + 1 = \overline{\overline{\bigcup S_4}}$. Then Card$\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^{\circ} S_4 \subseteq S_6 \ \wedge \ \mathrm{conv}^{@} S_6 \subseteq \mathrm{conv}^{@} \bigcup S_4\} = 2$.

(44) Let given A_3. Suppose that

(i) K is a subdivision of the complex of $\{A_3\}$,

(ii) $\overline{\overline{A_3}} = n + 1$,

(iii) degree$(K) = n$, and

(iv) for every simplex S of $n - 1$ and K and for every X such that $X = \{S_6; S_6$ ranges over simplexes of n and K: $S \subseteq S_6\}$ holds if $\mathrm{conv}^{@} S$ meets Int A_3, then Card $X = 2$ and if $\mathrm{conv}^{@} S$ misses Int A_3, then Card $X = 1$.

Let S be a simplex of $n-1$ and BCS K and given X such that $X = \{S_6; S_6$ ranges over simplexes of n and BCS K : $S \subseteq S_6\}$. Then

(v) if $\mathrm{conv}^{@} S$ meets Int A_3, then Card $X = 2$, and

(vi) if $\mathrm{conv}^{@} S$ misses Int A_3, then Card $X = 1$.

(45) Let S be a simplex of $n - 1$ and BCS(k, the complex of $\{A_2\}$) such that $\overline{\overline{A_2}} = n + 1$ and $X = \{S_6; S_6$ ranges over simplexes of n and BCS(k, the complex of $\{A_2\}$): $S \subseteq S_6\}$. Then

(i) if $\mathrm{conv}^{@} S$ meets Int A_2, then Card $X = 2$, and

(ii) if $\mathrm{conv}^{@} S$ misses Int A_2, then Card $X = 1$.

6. The Main Theorem

In the sequel v is a vertex of BCS(k, the complex of $\{A_2\}$) and F is a function from Vertices BCS(k, the complex of $\{A_2\}$) into A_2.

The following two propositions are true:

(46) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in$ conv B holds $F(v) \in B$. Then there exists n such that Card$\{S; S$ ranges over

simplexes of $\overline{\overline{A_2}} - 1$ and $\mathrm{BCS}(k, \text{the complex of } \{A_2\})$: $F^\circ S = A_2\} = 2 \cdot n + 1$.

(47) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in \operatorname{conv} B$ holds $F(v) \in B$. Then there exists a simplex S of $\overline{\overline{A_2}} - 1$ and $\mathrm{BCS}(k, \text{the complex of } \{A_2\})$ such that $F^\circ S = A_2$.

References

[1] Broderick Arneson and Piotr Rudnicki. Recognizing chordal graphs: Lex BFS and MCS. *Formalized Mathematics*, 14(**4**):187–205, 2006, doi:10.2478/v10037-006-0022-z.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[5] Grzegorz Bancerek and Yasunari Shidama. Introduction to matroids. *Formalized Mathematics*, 16(**4**):325–332, 2008, doi:10.2478/v10037-008-0040-0.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] Roman Duda. *Wprowadzenie do topologii*. PWN, 1986.

[11] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Convex sets and convex combinations. *Formalized Mathematics*, 11(**1**):53–58, 2003.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[13] Adam Naumowicz. On Segre's product of partial line spaces. *Formalized Mathematics*, 9(**2**):383–390, 2001.

[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[16] Karol Pąk. Affine independence in vector spaces. *Formalized Mathematics*, 18(**1**):87–93, 2010, doi: 10.2478/v10037-010-0012-z.

[17] Karol Pąk. Abstract simplicial complexes. *Formalized Mathematics*, 18(**1**):95–106, 2010, doi: 10.2478/v10037-010-0013-y.

[18] Karol Pąk. The geometric interior in real linear spaces. *Formalized Mathematics*, 18(**3**):185–188, 2010, doi: 10.2478/v10037-010-0021-y.

[19] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[23] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 9, 2010