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Sperner's Lemma

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Summary. In this article we introduce and prove properties of simplicial complexes in real linear spaces which are necessary to formulate Sperner's lemma. The lemma states that for a function f, which for an arbitrary vertex v of the barycentric subdivision $\mathcal{B}$ of simplex $\mathcal{K}$ assigns some vertex from a face of $\mathcal{K}$ which contains v, we can find a simplex S of $\mathcal{B}$ which satisfies $f(S) = \mathcal{K}$ (see [10]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [11], [19], [9], [6], [7], [1], [5], [3], [4], [13], [15], [12], [22], [23], [16], [18], [20], [14], [17], [21], and [8].

1. Preliminaries

We follow the rules: x, y, X denote sets and n, k denote natural numbers.

The following two propositions are true:

(1) Let R be a binary relation and C be a cardinal number. If for every x such that $x \in X$ holds $\mathrm{Card}(R^{\circ}x) = C$, then $\mathrm{Card}\,R = \mathrm{Card}(R{\restriction}(\mathrm{dom}\,R \setminus X)) + C \cdot \mathrm{Card}\,X$.

(2) Let Y be a non empty finite set. Suppose $\mathrm{Card}\,X = \overline{\overline{Y}} + 1$. Let f be a function from X into Y. Suppose f is onto. Then there exists y such that $y \in Y$ and $\mathrm{Card}(f^{-1}(\{y\})) = 2$ and for every x such that $x \in Y$ and $x \neq y$ holds $\mathrm{Card}(f^{-1}(\{x\})) = 1$.

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Let X be a 1-sorted structure. A simplicial complex structure of X is a simplicial complex structure of the carrier of X. A simplicial complex of X is a simplicial complex of the carrier of X.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a subset of K. The functor ${}^{@}A$ yielding a subset of X is defined by:

(Def. 1) ${}^{@}A = A$.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a family of subsets of K. The functor ${}^{@}A$ yielding a family of subsets of X is defined by:

(Def. 2) ${}^{@}A = A$.

We now state the proposition

(3) Let X be a 1-sorted structure and K be a subset-closed simplicial complex structure of X. Suppose K is total. Let S be a finite subset of K. Suppose S is simplex-like. Then the complex of $\{{}^{@}S\}$ is a subsimplicial complex of K.

2. The Area of an Abstract Simplicial Complex

For simplicity, we adopt the following rules: R_1 denotes a non empty RLS structure, K_1, K_2, K_3 denote simplicial complex structures of R_1, V denotes a real linear space, and K_4 denotes a non void simplicial complex of V.

Let us consider R_1, K_1. The functor $|K_1|$ yields a subset of R_1 and is defined by:

(Def. 3) $x \in |K_1|$ iff there exists a subset A of K_1 such that A is simplex-like and $x \in \operatorname{conv}{}^{@}A$.

One can prove the following propositions:

(4) If the topology of $K_2 \subseteq$ the topology of K_3, then $|K_2| \subseteq |K_3|$.

(5) For every subset A of K_1 such that A is simplex-like holds $\operatorname{conv}{}^{@}A \subseteq |K_1|$.

(6) Let K be a subset-closed simplicial complex structure of V. Then $x \in |K|$ if and only if there exists a subset A of K such that A is simplex-like and $x \in \operatorname{Int}({}^{@}A)$.

(7) $|K_1|$ is empty iff K_1 is empty-membered.

(8) For every subset A of R_1 holds $|$the complex of $\{A\}| = \operatorname{conv} A$.

(9) For all families A, B of subsets of R_1 holds $|$the complex of $A \cup B| = |$the complex of $A| \cup |$the complex of $B|$.

3. The Subdivision of a Simplicial Complex

Let us consider R_1, K_1. A simplicial complex structure of R_1 is said to be a subdivision structure of K_1 if it satisfies the conditions (Def. 4).

(Def. 4)(i) $|K_1| \subseteq |\mathrm{it}|$, and

(ii) for every subset A of it such that A is simplex-like there exists a subset B of K_1 such that B is simplex-like and $\mathrm{conv}^{@} A \subseteq \mathrm{conv}^{@} B$.

The following proposition is true

(10) For every subdivision structure P of K_1 holds $|K_1| = |P|$.

Let us consider R_1 and let K_1 be a simplicial complex structure of R_1 with a non-empty element. Observe that every subdivision structure of K_1 has a non-empty element.

We now state four propositions:

(11) K_1 is a subdivision structure of K_1.

(12) The complex of the topology of K_1 is a subdivision structure of K_1.

(13) Let K be a subset-closed simplicial complex structure of V and S_1 be a family of subsets of K. Suppose $S_1 = \mathrm{SubFin}(\text{the topology of } K)$. Then the complex of S_1 is a subdivision structure of K.

(14) For every subdivision structure P_1 of K_1 holds every subdivision structure of P_1 is a subdivision structure of K_1.

Let us consider V and let K be a simplicial complex structure of V. Note that there exists a subdivision structure of K which is finite-membered and subset-closed.

Let us consider V and let K be a simplicial complex structure of V. A subdivision of K is a finite-membered subset-closed subdivision structure of K.

We now state the proposition

(15) Let K be a simplicial complex of V with empty element. Suppose $|K| \subseteq \Omega_K$. Let B be a function from $2_{+}^{\text{the carrier of } V}$ into the carrier of V. Suppose that for every simplex S of K such that S is non empty holds $B(S) \in \mathrm{conv}^{@} S$. Then $\mathrm{subdivision}(B, K)$ is a subdivision structure of K.

Let us consider V, K_4. One can verify that there exists a subdivision of K_4 which is non void.

4. The Barycentric Subdivision

Let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}\, K_4$ yields a non void subdivision of K_4 and is defined by:

(Def. 5) $\mathrm{BCS}\, K_4 = \mathrm{subdivision}(\text{the center of mass of } V, K_4)$.

Let us consider n and let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}(n, K_4)$ yields a non void subdivision of K_4 and is defined by:

(Def. 6) $\mathrm{BCS}(n, K_4) = \mathrm{subdivision}(n, \text{the center of mass of } V, K_4)$.

Next we state several propositions:

(16) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(0, K_4) = K_4$.

(17) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(1, K_4) = \mathrm{BCS}\, K_4$.

(18) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\Omega_{\mathrm{BCS}(n,K_4)} = \Omega_{(K_4)}$.

(19) If $|K_4| \subseteq \Omega_{(K_4)}$, then $|\mathrm{BCS}(n, K_4)| = |K_4|$.

(20) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(n+1, K_4) = \mathrm{BCS}\,\mathrm{BCS}(n, K_4)$.

(21) If $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}\, K_4$.

(22) If $n > 0$ and $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}(n, K_4)$.

(23) Let S_2 be a non void subsimplicial complex of K_4. If $|K_4| \subseteq \Omega_{(K_4)}$ and $|S_2| \subseteq \Omega_{(S_2)}$, then $\mathrm{BCS}(n, S_2)$ is a subsimplicial complex of $\mathrm{BCS}(n, K_4)$.

(24) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{Vertices}\, K_4 \subseteq \mathrm{Vertices}\,\mathrm{BCS}(n, K_4)$.

Let us consider n, V and let K be a non void total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is total.

Let us consider n, V and let K be a non void finite-vertices total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is finite-vertices.

5. Selected Properties of Simplicial Complexes

Let us consider V and let K be a simplicial complex structure of V. We say that K is affinely-independent if and only if:

(Def. 7) For every subset A of K such that A is simplex-like holds ${}^{@}A$ is affinely-independent.

Let us consider R_1, K_1. We say that K_1 is simplex-join-closed if and only if:

(Def. 8) For all subsets A, B of K_1 such that A is simplex-like and B is simplex-like holds $\mathrm{conv}^{@}A \cap \mathrm{conv}^{@}B = \mathrm{conv}^{@}A \cap B$.

Let us consider V. Note that every simplicial complex structure of V which is empty-membered is also affinely-independent. Let F be an affinely-independent family of subsets of V. Observe that the complex of F is affinely-independent.

Let us consider R_1. One can verify that every simplicial complex structure of R_1 which is empty-membered is also simplex-join-closed.

Let us consider V and let I be an affinely-independent subset of V. One can check that the complex of $\{I\}$ is simplex-join-closed.

Let us consider V. One can check that there exists a subset of V which is non empty, trivial, and affinely-independent.

Let us consider V. One can check that there exists a simplicial complex of V which is finite-vertices, affinely-independent, simplex-join-closed, and total and has a non-empty element.

Let us consider V and let K be an affinely-independent simplicial complex structure of V. One can verify that every subsimplicial complex of K is affinely-independent.

Let us consider V and let K be a simplex-join-closed simplicial complex structure of V. One can check that every subsimplicial complex of K is simplex-join-closed.

Next we state the proposition

(25) Let K be a subset-closed simplicial complex structure of V. Then K is simplex-join-closed if and only if for all subsets A, B of K such that A is simplex-like and B is simplex-like and $\mathrm{Int}(^{@}A)$ meets $\mathrm{Int}(^{@}B)$ holds $A = B$.

For simplicity, we follow the rules: K_5 is a simplex-join-closed simplicial complex of V, A_1, B_1 are subsets of K_5, K_6 is a non void affinely-independent simplicial complex of V, K_7 is a non void affinely-independent simplex-join-closed simplicial complex of V, and K is a non void affinely-independent simplex-join-closed total simplicial complex of V.

Let us consider V, K_6 and let S be a simplex of K_6. Note that $^{@}S$ is affinely-independent.

One can prove the following propositions:

(26) If A_1 is simplex-like and B_1 is simplex-like and $\mathrm{Int}(^{@}A_1)$ meets $\mathrm{conv}^{@}B_1$, then $A_1 \subseteq B_1$.

(27) If A_1 is simplex-like and $^{@}A_1$ is affinely-independent and B_1 is simplex-like, then $\mathrm{Int}(^{@}A_1) \subseteq \mathrm{conv}^{@}B_1$ iff $A_1 \subseteq B_1$.

(28) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{BCS}\,K_6$ is affinely-independent.

Let us consider V and let K_6 be a non void affinely-independent total simplicial complex of V. Observe that $\mathrm{BCS}\,K_6$ is affinely-independent. Let us consider n. Observe that $\mathrm{BCS}(n, K_6)$ is affinely-independent.

Let us consider V, K_7. One can verify that (the center of mass of V)$\restriction$the topology of K_7 is one-to-one.

We now state the proposition

(29) If $|K_7| \subseteq \Omega_{(K_7)}$, then $\mathrm{BCS}\,K_7$ is simplex-join-closed.

Let us consider V, K. Note that $\mathrm{BCS}\,K$ is simplex-join-closed. Let us consider n. Observe that $\mathrm{BCS}(n, K)$ is simplex-join-closed.

The following four propositions are true:

(30) Suppose $|K_4| \subseteq \Omega_{(K_4)}$ and for every n such that $n \leq \mathrm{degree}(K_4)$ there exists a simplex S of K_4 such that $\overline{\overline{S}} = n + 1$ and $^{@}S$ is affinely-independent. Then $\mathrm{degree}(K_4) = \mathrm{degree}(\mathrm{BCS}\,K_4)$.

(31) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}\,K_6)$.

(32) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}(n, K_6))$.

(33) Let S be a simplex-like family of subsets of K_7. If S has non empty elements, then $\operatorname{Card} S = \operatorname{Card}((\text{the center of mass of } V)^\circ S)$.

For simplicity, we adopt the following convention: A_2 denotes a finite affinely-independent subset of V, A_3, B_2 denote finite subsets of V, B denotes a subset of V, S, T denote finite families of subsets of V, S_3 denotes a $\subseteq$-linear finite finite-membered family of subsets of V, S_4, T_1 denote finite simplex-like families of subsets of K, and A_4 denotes a simplex of K.

The following propositions are true:

(34) Let S_6, S_5 be simplex-like families of subsets of K_7. Suppose that
(i) $|K_7| \subseteq \Omega_{(K_7)}$,
(ii) S_6 has non empty elements,
(iii) (the center of mass of $V)^\circ S_5$ is a simplex of $\operatorname{BCS} K_7$, and
(iv) (the center of mass of $V)^\circ S_6 \subseteq$ (the center of mass of $V)^\circ S_5$.
Then $S_6 \subseteq S_5$ and S_5 is $\subseteq$-linear.

(35) Suppose S has non empty elements and $\bigcup S \subseteq A_2$ and $\overline{\overline{S}} + n + 1 \leq \overline{\overline{A_2}}$. Then the following statements are equivalent
(i) B_2 is a simplex of $n + \overline{\overline{S}}$ and BCS (the complex of $\{A_2\}$) and (the center of mass of $V)^\circ S \subseteq B_2$,
(ii) there exists T such that T misses S and $T \cup S$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T}} = n + 1$ and $\bigcup T \subseteq A_2$ and $B_2 =$ (the center of mass of $V)^\circ S \cup$ (the center of mass of $V)^\circ T$.

(36) Suppose S_3 has non empty elements and $\bigcup S_3 \subseteq A_2$. Then the following statements are equivalent
(i) (the center of mass of $V)^\circ S_3$ is a simplex of $\overline{\overline{\bigcup S_3}} - 1$ and BCS (the complex of $\{A_2\}$),
(ii) for every n such that $0 < n \leq \overline{\overline{\bigcup S_3}}$ there exists x such that $x \in S_3$ and $\operatorname{Card} x = n$.

(37) Let given S. Suppose S is $\subseteq$-linear and has non empty elements and $\overline{\overline{S}} = \operatorname{Card} \bigcup S$. Let given A_3, B_2. Suppose A_3 is non empty and A_3 misses $\bigcup S$ and $\bigcup S \cup A_3$ is affinely-independent and $\bigcup S \cup A_3 \subseteq B_2$. Then (the center of mass of $V)^\circ S \cup$ (the center of mass of $V)^\circ \{\bigcup S \cup A_3\}$ is a simplex of $\overline{\overline{S}}$ and BCS (the complex of $\{B_2\}$).

(38) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} = \overline{\overline{\bigcup S_3}}$. Let v be an element of V. Suppose $v \notin \bigcup S_3$ and $\bigcup S_3 \cup \{v\}$ is affinely-independent. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3 \cup \{v\}\}$): (the center of mass of $V)^\circ S_3 \subseteq S_6\} = \{$(the center of mass of $V)^\circ S_3 \cup$ (the center of mass of $V)^\circ \{\bigcup S_3 \cup \{v\}\}\}$.

(39) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} + 1 = \overline{\overline{\bigcup S_3}}$ and $\bigcup S_3$ is affinely-independent. Then $\operatorname{Card}\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3\}$): (the center of mass of $V)^\circ S_3 \subseteq S_6\} = 2$.

(40) Suppose A_2 is a simplex of K. Then B is a simplex of BCS (the complex of $\{A_2\}$) if and only if B is a simplex of BCS K and conv $B \subseteq$ conv A_2.

(41) Suppose S_4 has non empty elements and $\overline{\overline{S_4}} + n \leq$ degree(K). Then the following statements are equivalent

(i) A_3 is a simplex of $n + \overline{\overline{S_4}}$ and BCS K and (the center of mass of $V)^\circ S_4 \subseteq A_3$,

(ii) there exists T_1 such that T_1 misses S_4 and $T_1 \cup S_4$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T_1}} = n + 1$ and $A_3 =$ (the center of mass of $V)^\circ S_4 \cup$ (the center of mass of $V)^\circ T_1$.

(42) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} = \overline{\overline{\bigcup S_4}}$ and $\bigcup S_4 \subseteq A_4$ and $\overline{\overline{A_4}} = \overline{\overline{S_4}} + 1$. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^\circ S_4 \subseteq S_6 \wedge \mathrm{conv}^@ S_6 \subseteq \mathrm{conv}^@ A_4\} =$ $\{$(the center of mass of $V)^\circ S_4 \cup$ (the center of mass of $V)^\circ\{A_4\}\}$.

(43) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} + 1 = \overline{\overline{\bigcup S_4}}$. Then Card$\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^\circ S_4 \subseteq S_6 \ \wedge\ \mathrm{conv}^@ S_6 \subseteq \mathrm{conv}^@ \bigcup S_4\} = 2$.

(44) Let given A_3. Suppose that

(i) K is a subdivision of the complex of $\{A_3\}$,

(ii) $\overline{\overline{A_3}} = n + 1$,

(iii) degree$(K) = n$, and

(iv) for every simplex S of $n - 1$ and K and for every X such that $X = \{S_6; S_6$ ranges over simplexes of n and K: $S \subseteq S_6\}$ holds if $\mathrm{conv}^@ S$ meets Int A_3, then Card $X = 2$ and if $\mathrm{conv}^@ S$ misses Int A_3, then Card $X = 1$.

Let S be a simplex of $n-1$ and BCS K and given X such that $X = \{S_6; S_6$ ranges over simplexes of n and BCS K : $S \subseteq S_6\}$. Then

(v) if $\mathrm{conv}^@ S$ meets Int A_3, then Card $X = 2$, and

(vi) if $\mathrm{conv}^@ S$ misses Int A_3, then Card $X = 1$.

(45) Let S be a simplex of $n - 1$ and BCS$(k$, the complex of $\{A_2\})$ such that $\overline{\overline{A_2}} = n + 1$ and $X = \{S_6; S_6$ ranges over simplexes of n and BCS$(k$, the complex of $\{A_2\})$: $S \subseteq S_6\}$. Then

(i) if $\mathrm{conv}^@ S$ meets Int A_2, then Card $X = 2$, and

(ii) if $\mathrm{conv}^@ S$ misses Int A_2, then Card $X = 1$.

6. The Main Theorem

In the sequel v is a vertex of BCS$(k$, the complex of $\{A_2\})$ and F is a function from Vertices BCS$(k$, the complex of $\{A_2\})$ into A_2.

The following two propositions are true:

(46) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in$ conv B holds $F(v) \in B$. Then there exists n such that Card$\{S; S$ ranges over

simplexes of $\overline{\overline{A_2}} - 1$ and BCS(k, the complex of $\{A_2\}$): $F^\circ S = A_2\} = 2 \cdot n + 1$.

(47) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in \operatorname{conv} B$ holds $F(v) \in B$. Then there exists a simplex S of $\overline{\overline{A_2}} - 1$ and BCS(k, the complex of $\{A_2\}$) such that $F^\circ S = A_2$.

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