

Differentiation of Vector-Valued Functions on n -Dimensional Real Normed Linear Spaces

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Summary. In this article, we define and develop differentiation of vector-valued functions on n -dimensional real normed linear spaces (refer to [16] and [17]).

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The papers [8], [14], [2], [3], [4], [5], [13], [18], [1], [12], [6], [10], [15], [11], [9], [21], [19], [20], and [7] provide the terminology and notation for this paper.

1. THE BASIC PROPERTIES OF DIFFERENTIATION OF FUNCTIONS FROM \mathcal{R}^m TO \mathcal{R}^n

In this paper i, n, m are elements of \mathbb{N} .

The following propositions are true:

- (1) Let f be a set. Then f is a partial function from \mathcal{R}^m to \mathcal{R}^n if and only if f is a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$.
- (2) Let n, m be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of \mathcal{R}^m , and y be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $x = y$. Then f is differentiable in x if and only if g is differentiable in y .

- (3) Let n, m be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of \mathcal{R}^m , and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $f = g$ and $x = y$ and f is differentiable in x , then $f'(x) = g'(y)$.
- (4) Let f_1, f_2 be partial functions from \mathcal{R}^m to \mathcal{R}^n and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.
- (5) Let f_1, f_2 be partial functions from \mathcal{R}^m to \mathcal{R}^n and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.
- (6) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a be a real number. If $f = g$, then $a f = a g$.
- (7) Let f_1, f_2 be functions from \mathcal{R}^m into \mathcal{R}^n and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.
- (8) Let f_1, f_2 be functions from \mathcal{R}^m into \mathcal{R}^n and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.
- (9) Let f be a function from \mathcal{R}^m into \mathcal{R}^n , g be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and r be a real number. If $f = g$, then $r f = r \cdot g$.
- (10) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x be an element of \mathcal{R}^m . Suppose f is differentiable in x . Then $f'(x)$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let n, m be natural numbers and let I_1 be a function from \mathcal{R}^m into \mathcal{R}^n . We say that I_1 is additive if and only if:

(Def. 1) For all elements x, y of \mathcal{R}^m holds $I_1(x + y) = I_1(x) + I_1(y)$.

We say that I_1 is homogeneous if and only if:

(Def. 2) For every element x of \mathcal{R}^m and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

The following three propositions are true:

- (11) For every function I_1 from \mathcal{R}^m into \mathcal{R}^n such that I_1 is additive holds $I_1(\underbrace{\langle 0, \dots, 0 \rangle}_m) = \underbrace{\langle 0, \dots, 0 \rangle}_n$.
- (12) Let f be a function from \mathcal{R}^m into \mathcal{R}^n and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is additive iff g is additive.
- (13) Let f be a function from \mathcal{R}^m into \mathcal{R}^n and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is homogeneous iff g is homogeneous.

Let n, m be natural numbers. One can verify that the function $\mathcal{R}^m \mapsto \underbrace{\langle 0, \dots, 0 \rangle}_n$ is additive and homogeneous.

Let n, m be natural numbers. Note that there exists a function from \mathcal{R}^m into \mathcal{R}^n which is additive and homogeneous.

Let m, n be natural numbers. A linear operator from m into n is defined by an additive homogeneous function from \mathcal{R}^m into \mathcal{R}^n .

We now state the proposition

- (14) Let f be a set. Then f is a linear operator from m into n if and only if f is a linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let m, n be natural numbers, let I_1 be a function from \mathcal{R}^m into \mathcal{R}^n , and let x be an element of \mathcal{R}^m . Then $I_1(x)$ is an element of \mathcal{R}^n .

Let m, n be natural numbers and let I_1 be a function from \mathcal{R}^m into \mathcal{R}^n . We say that I_1 is bounded if and only if:

- (Def. 3) There exists a real number K such that $0 \leq K$ and for every element x of \mathcal{R}^m holds $|I_1(x)| \leq K \cdot |x|$.

Next we state three propositions:

- (15) Let x_1, y_1 be finite sequences of elements of \mathcal{R}^m . Suppose $\text{len } x_1 = \text{len } y_1 + 1$ and $x_1 \upharpoonright \text{len } y_1 = y_1$. Then there exists an element v of \mathcal{R}^m such that $v = x_1(\text{len } x_1)$ and $\sum x_1 = \sum y_1 + v$.
- (16) Let f be a linear operator from m into n , x_1 be a finite sequence of elements of \mathcal{R}^m , and y_1 be a finite sequence of elements of \mathcal{R}^n . Suppose $\text{len } x_1 = \text{len } y_1$ and for every element i of \mathbb{N} such that $i \in \text{dom } x_1$ holds $y_1(i) = f(x_1(i))$. Then $\sum y_1 = f(\sum x_1)$.
- (17) Let x_1 be a finite sequence of elements of \mathcal{R}^m and y_1 be a finite sequence of elements of \mathbb{R} . Suppose $\text{len } x_1 = \text{len } y_1$ and for every element i of \mathbb{N} such that $i \in \text{dom } x_1$ there exists an element v of \mathcal{R}^m such that $v = x_1(i)$ and $y_1(i) = |v|$. Then $|\sum x_1| \leq \sum y_1$.

Let m, n be natural numbers. Note that every linear operator from m into n is bounded.

Let us consider m, n . Observe that every linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is bounded.

Next we state several propositions:

- (18) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x be an element of \mathcal{R}^m . Suppose f is differentiable in x . Then $f'(x)$ is a linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.
- (19) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x be an element of \mathcal{R}^m . Suppose f is differentiable in x . Then $f'(x)$ is a linear operator from m into n .
- (20) Let n, m be non empty elements of \mathbb{N} , g_1, g_2 be partial functions from

\mathcal{R}^m to \mathcal{R}^n , and y_0 be an element of \mathcal{R}^m . Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0 . Then $g_1 + g_2$ is differentiable in y_0 and $(g_1 + g_2)'(y_0) = g_1'(y_0) + g_2'(y_0)$.

(21) Let n, m be non empty elements of \mathbb{N} , g_1, g_2 be partial functions from \mathcal{R}^m to \mathcal{R}^n , and y_0 be an element of \mathcal{R}^m . Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0 . Then $g_1 - g_2$ is differentiable in y_0 and $(g_1 - g_2)'(y_0) = g_1'(y_0) - g_2'(y_0)$.

(22) Let n, m be non empty elements of \mathbb{N} , g be a partial function from \mathcal{R}^m to \mathcal{R}^n , y_0 be an element of \mathcal{R}^m , and r be a real number. Suppose g is differentiable in y_0 . Then rg is differentiable in y_0 and $(rg)'(y_0) = r g'(y_0)$.

(23) Let x_0 be an element of \mathcal{R}^m , y_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and r be a real number. Suppose $x_0 = y_0$. Then $\{y \in \mathcal{R}^m: |y - x_0| < r\} = \{z; z \text{ ranges over points of } \langle \mathcal{E}^m, \|\cdot\| \rangle: \|z - y_0\| < r\}$.

(24) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , x_0 be an element of \mathcal{R}^m , and L, R be functions from \mathcal{R}^m into \mathcal{R}^n . Suppose that

- (i) L is a linear operator from m into n , and
- (ii) there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m: |y - x_0| < r_0\} \subseteq \text{dom } f$ and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of \mathcal{R}^m and for every element w of \mathcal{R}^n such that $z \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$ and $|z| < d$ and $w = R(z)$

holds $|z|^{-1} \cdot |w| < r$ and for every element x of \mathcal{R}^m such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Then f is differentiable in x_0 and $f'(x_0) = L$.

(25) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x_0 be an element of \mathcal{R}^m . Then f is differentiable in x_0 if and only if there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m: |y - x_0| < r_0\} \subseteq \text{dom } f$ and there exist functions L, R from \mathcal{R}^m into \mathcal{R}^n such that L is a linear operator from m into n and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of \mathcal{R}^m and for every element w of \mathcal{R}^n such that $z \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$

and $|z| < d$ and $w = R(z)$ holds $|z|^{-1} \cdot |w| < r$ and for every element x of \mathcal{R}^m such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

2. DIFFERENTIATION OF FUNCTIONS FROM NORMED LINEAR SPACES \mathcal{R}^m TO
NORMED LINEAR SPACES \mathcal{R}^n

One can prove the following propositions:

- (26) For all points y_2, y_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\text{Proj}(i, n))(y_2 + y_3) = (\text{Proj}(i, n))(y_2) + (\text{Proj}(i, n))(y_3)$.
- (27) For every point y_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every real number r holds $(\text{Proj}(i, n))(r \cdot y_2) = r \cdot (\text{Proj}(i, n))(y_2)$.
- (28) Let m, n be non empty elements of \mathbb{N} , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and i be an element of \mathbb{N} . Suppose $1 \leq i \leq n$ and g is differentiable in x_0 . Then $\text{Proj}(i, n) \cdot g$ is differentiable in x_0 and $\text{Proj}(i, n) \cdot g'(x_0) = (\text{Proj}(i, n) \cdot g)'(x_0)$.
- (29) Let m, n be non empty elements of \mathbb{N} , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Then g is differentiable in x_0 if and only if for every element i of \mathbb{N} such that $1 \leq i \leq n$ holds $\text{Proj}(i, n) \cdot g$ is differentiable in x_0 .

Let X be a set, let n, m be non empty elements of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathcal{R}^n . We say that f is differentiable on X if and only if:

- (Def. 4) $X \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in X$ holds $f|_X$ is differentiable in x .

The following four propositions are true:

- (30) Let X be a set, m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$. Then f is differentiable on X if and only if g is differentiable on X .
- (31) Let X be a set, m, n be non empty elements of \mathbb{N} , and f be a partial function from \mathcal{R}^m to \mathcal{R}^n . If f is differentiable on X , then X is a subset of \mathcal{R}^m .
- (32) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and Z be a subset of \mathcal{R}^m . Given a subset Z_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $Z = Z_0$ and Z_0 is open. Then f is differentiable on Z if and only if the following conditions are satisfied:
- (i) $Z \subseteq \text{dom } f$, and
 - (ii) for every element x of \mathcal{R}^m such that $x \in Z$ holds f is differentiable in x .
- (33) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and Z be a subset of \mathcal{R}^m . Suppose f is differentiable on Z . Then there exists a subset Z_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $Z = Z_0$ and Z_0 is open.

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